DOWLING GEOMETRIES ARE LINE CLOSED: A SHORT PROOF

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Abstract. We use gain graphs to provide a short direct proof that Dowling geometries are line closed.

The rank n Dowling geometry (or "matroid") $Q_n(\mathfrak{G})$ of a group $\mathfrak{G}$ was introduced by Dowling in [3]. A set S in a matroid is *line closed* if, whenever $x, y \in S$, the closure $\overline{xy} \subseteq S$. A matroid is called *line closed* if every line-closed subset is a flat. We present a short direct proof, employing the machinery of gain graphs [6], of the following observation of Bogart and Bonin [1]:

Theorem. *$Q_n(\mathfrak{G})$ is line closed.*

I do not claim a remarkable improvement over previous approaches, just a different kind of proof. Halsey [4] has given a short proof of a more general theorem: all supersolvable matroids are line closed. Since, as Dowling showed in another short proof, $Q_n(\mathfrak{G})$ is supersolvable, it is line closed. I have not studied Halsey's proof. Bogart and Bonin's purpose was not to prove the line closure of Dowling geometries so much as to use line closure as a foundation for developing Dowling geometries.

$Q_n(\mathfrak{G})$ can be defined graph-theoretically as the bias matroid $G(\mathfrak{G}K_n^\bullet)$ of a complete $\mathfrak{G}$-gain graph of order n, $\mathfrak{G}K_n^\bullet$, constructed as follows: Replace each edge vw of K_n by one edge (vw, g) (with the same endpoints) for each element $g \in \mathfrak{G}$ and add a half edge at each vertex. We say (vw, g) is a *link* with *gain* g from v to w and we identify it with (wv, g^{-1}), which we regard as the same edge in the opposite direction; that is, the gain of an edge is inverted by reversing direction. A half edge is an edge with only one end. We call a circle C (the edge set of a simple closed path) *balanced* if the product of the gains of its edges, taken in order around C and in the direction of traversal, equals the group identity. An edge set $S \subseteq E = E(\mathfrak{G}K_n^\bullet)$ is *balanced* if every circle in it is balanced and it contains no half edge. The key point is the description of flats and the closure operation of $G(\mathfrak{G}K_n^\bullet)$, from [6, Thm. 2.1]. For an edge set S (considered as a spanning subgraph (V, S)), we call its unbalanced connected components $S_{01}, \dots, S_{0k}$ and the balanced components $S_1, \dots, S_l$. (Possibly $k = 0$ or $l = 0$ or both.) We write $S_0 = S_{01} \cup \dots \cup S_{0k}$. We further write $X_i = V(S_i)$. When S is a flat, each S_i is a balanced complete subgraph of $\mathfrak{G}K_n^\bullet$ and $S_0 = E{:}X_0$, where $E{:}X$ denotes the edge set induced by a vertex set X. Thus for general S, $\overline{S} = E{:}X_0 \cup A_1 \cup \dots \cup A_l$ where each A_i, for $i > 0$, is a balanced complete subgraph on vertex set X_i, determined uniquely by containing S_i.

The graphic representation of $Q_n(\mathfrak{G})$ was known in partial form to Dowling, though not published by him; it was first published, again in partial form, by Doubilet–Rota–Stanley

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in [2, Sect. 5.3], and more completely, as part of a general theory of gain graphs and bias matroids, by Zaslavsky in, e.g., [5, 6] and [7, Ex. 5.7].

Lemma 1. *A balanced line-closed set S is closed.*

Proof. If S is connected, the matroid restricted to $\overline{S}$ is complete-graphic. This is known (and easily proved) to be line closed; hence S is closed. From this the lemma follows immediately. $\square$

Lemma 2. *An unbalanced line-closed set S is induced.*

Proof. First, let S be connected. We show that $\overline{S} = E{:}X_0$. S contains a spanning tree T, whose closure $\overline{T}$ is a balanced complete subgraph. By Lemma 1, $S \supseteq \overline{T}$. Since S is unbalanced, $S \supset \overline{T}$.

If S contains a half edge e on a vertex v, let $f \in \overline{T}$ be a link incident with v and let w be the other endpoint of f. Since $\{e, f\}$ is connected and unbalanced, $\overline{ef} = E{:}\{v, w\}$. Thus S, being line closed, contains the half edges at all vertices in X_0 as well as all links in $E{:}X_0$ incident to v. Now applying the same argument to all vertices $w \neq v$, we conclude that $S \supseteq E{:}X = \overline{S}$. Thus $S = E{:}X_0$.

Otherwise, S contains a link e parallel to a link $f \in \overline{T}$. Then $\overline{ef} = E{:}V(e)$. Therefore S, being line closed, contains the half edge at a vertex of e. It follows from the previous paragraph that $S = \overline{S}$.

Next, suppose that S is a union of unbalanced components $S_{01}, \dots, S_{0k}$ on vertex sets $X_{01}, \dots, X_{0k}$. Since S is line closed and S_{0i} is unbalanced, by the preceding argument $S_{0i} = E{:}X_{0i}$. An edge $e \in S \setminus (\bigcup_i S_{0i})$ has one endpoint $v \in X_{0i}$ and the other endpoint $w \in X_{0j}$ for some i, j. The half edges h_v, h_w at these vertices are in S. Therefore $e \in \overline{h_v h_w} \subseteq S$. Thus, $S = E{:}X_0$. $\square$

To prove the theorem, suppose S is arbitrary and $\overline{S} = E{:}X_0 \cup A_1 \cup \cdots \cup A_l$ as before. Being line closed, S contains $E{:}X_0$ by Lemma 2 and $A_1, \dots, A_l$ by Lemma 1. Therefore, $S = \overline{S}$. $\square$

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