GEOMETRIC LATTICES OF STRUCTURED PARTITIONS I: GAIN-GRAPHIC MATROIDS AND GROUP-VALUED PARTITIONS VERSION OF 1985 WITH MINOR REVISIONS

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Abstract. For certain matroids on the edges of a graph, digraph, or bidirected graph with group gains (edge labels from a group), the flats form intriguing new geometric lattices whose Whitney numbers have geometrical significance. In this chapter, the first of a series, we lay the theoretical foundation for the study of interesting special examples of graphical matroids and their lattices, including new general examples of geometric lattices based on groups and graphs, generalizing Dowling's lattices. (Indeed, fully half of our examples are join subsemilattices of Dowling lattices; the other half are a related but new type of geometric lattice.)

We list some cryptomorphisms of the bias, lift, and complete lift matroids of a gain graph and use them to describe in detail the lattices of flats in terms of subgraphs and in terms of group-valued partitions. We also give methods for calculating chromatic invariants. From the examples in subsequent chapters we abstract the idea of a structured partition, concerning which we raise several questions for further research.

Introduction

In this article and its successors [19] we describe several geometric lattices whose elements are "structured partitions"—partitions whose blocks carry a combinatorial structure —and which are based on matroids on the edge sets of undirected, directed, and bidirected graphs.

The matroids are particular types of matroids of biased graphs [17], more specifically of gain graphs (graphs whose edges are labelled by elements of a group), but some are not obviously so. They are the bicircular matroid of a graph, introduced by Simões-Pereira [**8**] and, in an infinitary version, Klee [5, pp. 145f.]; the poise and modular poise matroids of mixed graphs, introduced by Matthews for directed graphs [6]; the antidirection matroids of bidirected graphs, also first published for digraphs by Matthews [6]; and the general matroids of gain graphs [17], Part II]. Each of these examples is a kind of matroid we call a "bias" matroid, whose circuits are connected; closely related is the "(complete) lift" matroid, whose circuits are not necessarily connected.

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The structures on blocks of partitions that yield geometric lattices are diverse: spanning trees, bipartitions (partitions into at most two subblocks), compositions (ordered partitions into subblocks), sequenced decompositions, cyclically ordered partitions, and a special kind of partially ordered set we call a "tatter". All these are based on the group-valued partitions whose lattices were introduced by Dowling [3]. The main difference is that we consider examples where only some values or some partitions are allowed; thus we have many new examples related to Dowling's in much the same way the lattice of connected partitions of a graph is related to the lattice of partitions of the vertex set, and other new kinds of finite geometric lattices which also are join subsemilattices of Dowling's lattices. There is another significant difference: each structure gives not only Dowling-type lattices, whose elements are structured partial partitions of the ground set, but also modified lattices whose elements are structured and unstructured nonpartial partitions. (The former lattices are derived from the bias matroid, the latter from the lift.) Thus for each type of structure we have a pair of "universal" lattices (of which an example is a Dowling lattice and the lift analog) and a variety of interesting join subsemilattices, most based on graphs.

For instance a *bisected partition* is a partition whose blocks are each bipartitioned. The set of bisected partial partitions of an n-set is the universal "bias" lattice for bipartitioned blocks (and is better known in a slightly different presentation as Dowling's lattice of the two-element group); the corresponding "lift" lattice is the set of all bisected and ordinary partitions of an n-set. Given a graph Γ of order n, one associated geometric lattice is the set of bisected partial partitions of the n-set whose unpartitioned blocks are each Γ-connected. The spanning partitions of this type and all ordinary partitions of the n-set compose another geometric lattice. (See Chapter III.)

For each lattice we establish its meet and join, its interval structure (save in some very complicated cases), and as many of its "chromatic" invariants – characteristic polynomial, Möbius invariant, Whitney numbers – as we can evaluate, as well as the whole Möbius function when feasible. All this we believe to be new (except for the Dowling lattice results, most of which are Dowling's [3]). Our technique is to work in the underlying gain graph, using the matroid and chromatic theory developed in [17, Parts II and III]. We wish to emphasize, however, that we do not yet entirely know what kinds of structures on blocks yield geometric lattices similar to those treated herein, even though it appears that all such lattices are based on biased graphs (a result we hope to present in a later paper), nor what structures or combinations of them might give other kinds of lattices of structured partitions. We cannot even define a "structure" appropriately, for we do not have enough kinds of examples. We are simply exploiting gain-graphic methods to open a new field of investigation.

Several questions for study are raised at the end of this article. We hope that they and our work will help to stimulate the creation of a general theory of structured partitions.

In outline, this series consists of Chapter I, a general treatment of bias and (complete) lift matroids of graphs including cryptomorphisms, lattice structure, and chromatic invariants, and subsequent chapters in which various kinds of examples are developed. The bulk of Chapter I develops the lattice and invariant theory, working from [17, Parts II and III]. New materials are the treatment of the subgraph form of the lattices of the bias and complete lift matroids and especially the general analysis of lattices of restricted group-valued partitions (Section 3), also some results on chromatic invariants, and an odd lemma or proposition.

Chapter II applies the theory to construct and describe in detail various lattices of group-valued partitions over arbitrary groups. Almost all these results are new. Many of the evaluations of invariants are generalizations of results from [16] on signed graphs.

Chapters III et seq., which are independent of each other, are more combinatorial in flavor. Here we deal with purely combinatorial structures on partitions, again treating many examples to show the variety of lattices that arise and the power of our approach. Sometimes the approach is not powerful enough to evaluate the characteristic polynomial satisfactorily; then we reduce it to a combinatorial enumeration problem which we present to the reader. Chapter IV includes a construction, the biased union, by means of which we can combine different structures to get new lattices. It yields, for instance, lattices whose corresponding structures are "fat forests" (forest-linked cliques), or ordered m-tuples of partitions whose unions are acyclic hypergraphs. These examples are relatively difficult and very incompletely solved.

In all chapters we go to considerable trouble to compute not only the chromatic polynomial but an unfamiliar and fancy invariant, the Whitney number polynomial. This is explained by the fact that the latter tells us the numbers of faces of each dimension of an arrangement of hyperplanes. (See [17, Part IV] and [18].) Indeed, this work originated as an effort to count the faces of certain examples of the arrangements explored in [18], although it has grown well beyond that problem.

1. Notations and concepts

Here we define notations, objects, and concepts to be used throughout the series; most particularly, gain graphs and balance. Since we need a great deal of semistandard and nonstandard notation, this section is rather long. It is arranged by type of item, from sets, the simplest, to gain graphs, the most advanced.

1.1. **Sets and functions.** Some special sets are $[k, l] = \{k, k+1, \ldots, l\}$ if $k \leq l$ and $= \varnothing$ if $k > l$ for integers k, l, and $[n] = [1, n]$. The cardinality (or order) of a set S is $|S|$ or $\#S$. The power set of S is $\mathcal{P}(S)$. We commonly abbreviate $S \cup \{p\}$ by $S + p$. The disjoint union of sets is denoted by $X \mathbin{\dot\cup} Y$.

If f is a function and X a set, $f\big|_X$ denotes the restriction of f to $X \cap \operatorname{Dom} f$, where Dom means domain. The image is $\operatorname{Im} f = f(\operatorname{Dom} f)$.

Important functions include the falling factorial

$$(x)_k = x(x-1)\ldots(x-k+1) \text{ for } k \geq 0;$$

we interpret $(x)_0 = 1$. The semifactorial $m!!$ of a positive integer m is $m(m-2)(m-4)\cdots$, the product terminating at 2 or 1; and $0!! = 1$. The multinomial coefficient is

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}$$

if $n_1 + n_2 + \cdots + n_r = n$ and all $n_i \geq 0$; it is zero if any $n_i < 0$. Similarly, $\binom{x}{k} = (x)_k / k!$ if $k \geq 0$, $= 0$ if $k < 0$. The Stirling numbers of the first and second kinds are $s(n,k)$ and $S(n,k)$; they are zero if $k < 0$ or $k > n$.

1.2. **Posets and matroids.** Let L be a graded partially ordered set with zero; that is, a partially ordered set with a unique minimal element 0 (the *bottom* element) in which every maximal chain between each pair of elements has the same length. The rank of an element t is written $\operatorname{rk} t = \operatorname{rk}_L t$; the rank of L is $\operatorname{rk} L = \max\{\operatorname{rk} t : t \in L\}$. Let $L_i = \{t \in L : \operatorname{rk} t = i\}$. The *combinatorial Möbius function* $\mu(s,t) = \mu_L(s,t)$ (see e.g. [8]) is defined by $\mu(s,s) = 1$, $\mu(s,t) = 0$ if $s \not\leq t$, and for $s < t$ recursively by

$$\mu(s,t) = -\sum_{s \leq u < t} \mu(s,u).$$

The *doubly indexed Whitney numbers* of L of the *first kind* are

$$w_{ij} = w_{ij}(L) = \sum \{\mu(s,t) : s \in L_i,\ t \in L_j\}.$$

The *simply indexed* (or *ordinary*) Whitney numbers of the *first kind* are $w_j = w_{0j}$; those of the *second kind* are $W_j = w_{jj}$. The generating functions for numbers of the first kind are important in some applications; they are the *Whitney number polynomial*

$$w_L(x,y) = \sum\sum_{0 \leq i \leq j \leq \operatorname{rk} L} x^i w_{ij}(L) y^{\operatorname{rk} L - j}$$

and the *characteristic polynomial*

$$p_L(y) = w_L(0,y) = \sum_{j=0}^{\operatorname{rk} L} w_j(L) y^{\operatorname{rk} L - j}.$$

The *Möbius invariant*, defined only if L has not only a bottom element 0 but also a *top* (unique maximal) element 1, is $\mu(L) = \mu_L(0,1) = p_L(0)$.

If G is a matroid (we assume acquaintance with matroids as developed in [1, 7, 11, 12]), $\operatorname{Lat} G$ is its lattice of closed sets (or "flats"). The Whitney number polynomial of G is that of $\operatorname{Lat} G$; but the characteristic polynomial is $p_G(y) = p_{\operatorname{Lat} G}(y)$ if $\varnothing$ is closed, 0 otherwise, and the Möbius invariant of G is $\mu(G) = p_G(0)$, hence zero if $\varnothing$ is not closed. The Whitney numbers $w_{ij}(G)$ and $W_j(G)$ are those of $\operatorname{Lat} G$, but the $w_j(G)$ are defined by $p_G(y) = \sum y^{\operatorname{rk} G - j} w_j(G)$, hence equal $w_j(\operatorname{Lat} G)$ if $\varnothing$ is closed and zero otherwise.

1.3. **Partitions.** A *partition* π is a class of pairwise disjoint, nonempty sets, called the *blocks* of π. The *support* is the union of the blocks: $\operatorname{supp} \pi = \bigcup \pi$. A *[partial] partition of* V is a partition whose support is [a subset of] V; the set of [partial] partitions of V is Π_V [respectively, $\Pi_V^\dagger$] and that of an n-set is Π_n [respectively, $\Pi_n^\dagger$]. A partition of V is also said to *span* V. Special partitions are the *trivial* partition of V, $1_V = \{V\}$ (if $V \neq \varnothing$), the *total* partition 0_V whose blocks are singletons, and the unique partition of $\varnothing$, namely $\varnothing = 0_\varnothing$. We define $1_\varnothing = 0_\varnothing$. The *restriction* of π to a set S is

$$\pi|_S = \{B \cap S : B \in \pi \text{ and } B \cap S \neq \varnothing\}.$$

Let π_1 and π_2 be partitions, possibly of different sets. We say $\pi_1 \leq \pi_2$ (π_1 *refines* π_2) if every block of π_1 is either disjoint from $\operatorname{supp} \pi_2$ or lies within a block of π_2. Refinement is a partial ordering of Π_n and $\Pi_n^\dagger$; their bottom element is 0_V and their top elements are respectively 1_V and $0_\varnothing$. Notice that $\Pi_n^\dagger \cong \Pi_{n+1}$. The meet and join of π_1 and π_2 can be described as follows. Let $Y = (\operatorname{supp} \pi_1) \cap (\operatorname{supp} \pi_2)$ and $X_i = (\operatorname{supp} \pi_i) \setminus Y$. The *meet* is

$$\pi_1 \wedge \pi_2 = (\pi_1|_{X_1}) \cup (\pi_2|_{X_2}) \cup \{B_1 \cap B_2 : B_i \in \pi_i \text{ and } B_1 \cap B_2 \neq \varnothing\}$$

Let $\sim$ be the transitive closure of the relation "x and y lie in a common block of π_1 or π_2" on the set $(\operatorname{supp}\pi_1) \cup (\operatorname{supp}\pi_2)$. The *join* is

$$\pi_1 \vee \pi_2 = \{C : C \text{ is an equivalence class of } \sim \text{ and } C \subseteq Y\}.$$

The Whitney numbers of Π_n are well known: $W_j(\Pi_n) = S(n, n-j)$ and $w_j(\Pi_n) = s(n, n-j)$; in particular $\mu(\Pi_n) = (-1)^{n-1}(n-1)!$ [4].

1.4. **Graphs.** A *graph* Γ consists of a vertex set $V(\Gamma)$ and an edge set $E(\Gamma)$ (assumed disjoint from each other) and an endpoint mapping $\omega = \omega_\Gamma$ on $E(\Gamma)$ such that $\omega(e)$ is a multiset of at most two vertices, not necessarily distinct. (This definition allows multiple edges.) If $|\omega(e)| = 2$, say $\omega(e) = \{v, w\}$, we call e a *link* if $v \neq w$ and a *loop* if $v = w$. If $|\omega(e)| = 1$, e is a *half edge*. If $|\omega(e)| = 0$, e is a *loose edge*. An *ordinary graph* has no half or loose edges. Our notation for edges varies to suit the circumstances; often we write a link or loop as vw, or e_{vw}, to indicate its endpoints.

We may say: $\Delta = (X, S)$ is a graph. This means $V(\Delta) = X$ (also said: Δ *spans* X) and $E(\Delta) = S$; the endpoint mapping is here left implicit. Always, Γ will be a graph with vertex set V, hence of order n. Some special graphs are the complete graphs K_n of order n, more precisely K_X on vertex set X, including the null graph $K_0 = (\varnothing, \varnothing)$. A *simple* graph has only links and no multiple edges. A *full* graph has a half edge at every vertex. A *circle* (elementary cycle, circuit, polygon) is the (graph or) edge set of a simple closed path (of positive length). The set of circles of Γ is $\mathcal{C}(\Gamma)$.

Some operations on graphs: *Union* and *intersection* are performed on vertices and edges separately, thus $(X, S) \cup (Y, T) = (X \cup Y,\ S \cup T)$. (Here it is assumed that the endpoint mappings of the two graphs agree on the intersection.) The *join* $\Gamma_1 + \Gamma_2$ is the disjoint union $\Gamma_1 \cup \Gamma_2$ together with all edges vw for $v \in V(\Gamma_1)$ and $w \in V(\Gamma_2)$. By $\Gamma^\bullet$ we mean Γ with a half edge added at every vertex not possessing one. The *complement* of a simple graph Γ is denoted by $\bar{\Gamma}$ or Γ^c.

If $\Gamma = (V, E)$, the *induced* subgraph on $X \subseteq V$ is $\Gamma{:}X = (X, E{:}X)$, where $E{:}X = \{e \in E : \varnothing \neq \omega(e) \subseteq X\}$. A partial partition π of V induces $\Gamma{:}\pi = \bigcup\{\Gamma{:}B : B \in \pi\}$. A *subgraph* of Γ is any (X, S) with $X \subseteq V$ and $S \subseteq E{:}X$; it is *spanning* if $X = V$. We write $\Gamma|S$, the *restriction* to S, as an alternative to (V, S). A set $X \subseteq V$ is Γ-*connected* if $\Gamma{:}X$ is connected; a partial partition π is Γ-connected, or a Γ-*partition*, if each block is Γ-connected. We define

$$\begin{aligned}
\Pi(\Gamma) &= \{\pi \in \Pi_V : \pi \text{ is } \Gamma\text{-connected}\},\\
\Pi^\dagger(\Gamma) &= \{\pi \in \Pi_V^\dagger : \pi \text{ is } \Gamma\text{-connected and each component of}\\
&\qquad\qquad \Gamma{:}(\operatorname{supp}\pi)^c \text{ contains a half edge}\}.
\end{aligned}$$

These are equal if Γ has no half edges; on the other hand $\Pi^\dagger(\Gamma^\bullet)$ consists of all Γ-connected partial partitions of V. The top element of $\Pi(\Gamma)$ is $\pi(\Gamma)$, whose blocks are the vertex sets of the (connected) *components* of Γ; we write $c(\Gamma)$ for $|\pi(\Gamma)|$, the number of components. Some examples: $\Pi(K_n) = \Pi_n$, $\Pi^\dagger(K_n^\bullet) = \Pi_n^\dagger$.

The opposite of a subgraph induced by a partial partition is the *external* subgraph $\Gamma\langle\pi\rangle = (\operatorname{supp}\pi, E\langle\pi\rangle)$, where $E\langle\pi\rangle = \{e \in E : e \text{ has endpoints in two different blocks of } \pi\}$. For arbitrary $X, Y \subseteq V$ we define

$$E\langle X, Y\rangle = \{e \in E : e \text{ has one end in } X \text{ and the other in } Y\},$$

thus $\Gamma\langle X, Y\rangle = (X \cup Y, E\langle X, Y\rangle)$.

We can now define various kinds of *contraction*. The contraction Γ/S of Γ by $S \subseteq E$, where S contains no half edges, is $(\pi(V,S), E\setminus S)$ with endpoint mapping $\omega_{\Gamma/S}(e) = \{\bar{v} : v \in \omega(e)\}$, $\bar{v}$ denoting the block of $\pi(V,S)$ containing v. The contraction Γ/Δ by a subgraph Δ without half edges is $[\Gamma{:}V(\Delta)]/E(\Delta)$. The contraction Γ/π by a partial partition π is $(\pi, E\langle\pi\rangle)$ with endpoint mapping similar to $\omega_{\Gamma/S}$. Contraction by half edges is more complicated, being really a special case of the gain graph contraction to be treated shortly.

The *graphic* or *polygon matroid* $G(\Gamma)$ is the usual one if Γ is ordinary: the circuits are the circles. In general there are additional circuits of the form: two half edges joined by a simple path (of length possibly zero). We have $\Pi^\dagger(\Gamma) \cong \operatorname{Lat} G(\Gamma)$. (The graphic matroid of Γ is isomorphic to $G(\Gamma_0)$, where Γ_0 is Γ with an extra vertex v_0 and all half edges replace by links to v_0; cf. [14, Section 7A].)

Chromatic invariants of a finite ordinary graph Γ (of order n, say) are the chromatic polynomial $\chi_\Gamma(y)$; the *Whitney number polynomial*

$$\begin{aligned} w_\Gamma(x,y) &= \sum_{S\subseteq E} x^{\operatorname{rk} S}\chi_{\Gamma/S}(y) = \sum_{\pi\in\Pi(\Gamma)} x^{n-|\pi|}\chi_{\Gamma/\pi}(y) \\ &= \sum_{\kappa:V\to[y]} x^{\operatorname{rk} I(\kappa)}, \end{aligned}$$

where the summands for which S is not closed are zero and in the last summation y is a positive integer and $I(\kappa)$ is the set of edges improperly colored by the mapping κ; and the Whitney numbers $w_{ij}(\Gamma) =$ coefficient of $x^i y^{n-j}$ in $w_\Gamma(x,y)$, $w_j(\Gamma) =$ coefficient of y^{n-j} in $\chi_\Gamma(y)$ ($= w_{0j}(\Gamma)$ if Γ is loopless), and $W_j(\Gamma) = w_{jj}(\Gamma) = |\Pi(\Gamma)_j|$. By Rota's theorem [8], we have

$$\chi_\Gamma(y) = \begin{cases} y^{c(\Gamma)} p_{\operatorname{Lat} G(\Gamma)}(y) & \text{if } \Gamma \text{ is loopless,} \\ \\ 0 & \text{if } \Gamma \text{ has a loop,} \end{cases}$$

and consequently the similar formula

$$w_\Gamma(x,y) = y^{c(\Gamma)} w_{\operatorname{Lat} G(\Gamma)}(x,y) \text{ or } 0$$

for the Whitney number polynomial. One can thus evaluate the Möbius invariant, for instance, from the chromatic polynomial: $\mu(\Pi(\Gamma)) = \chi_\Gamma(y)/y^{c(\Gamma)}\big|_{y=0}$.

Some examples: $\chi_{K_n}(y) = (y)_n$ and

$$w_{K_n}(x,y) = \sum_{i=0}^{n} x^{n-i} S(n,i)(y)_i.$$

1.5. **Gain and biased graphs.** A *gain graph* Φ with *gain group* $\mathfrak{G}$ consists of an underlying graph $\|\Phi\|$, say $\Gamma = (V,E)$, and a *gain mapping*, which is a partial function $\varphi : E \to \mathfrak{G}$ defined on links and loops and satisfying $\varphi(e^{-1}) = \varphi(e)^{-1}$, where e^{-1} means e taken in the opposite direction. This is slightly vague; to be precise we take

$$\varphi : \{e; v, w) : e \in E,\ \omega(e) = \{v,w\}\} \to \mathfrak{G},$$

satisfying

$$\varphi(e; w, v) = \varphi(e; v, w)^{-1}$$

if e is a link. We abbreviate Φ by (V,E,φ) or $(V,E,\mathfrak{G},\varphi)$ or (Γ,φ), etc. We may also write $V(\Phi)$, $E(\Phi)$, etc. The *order* of Φ is $|V|$.

A path $C = v_0 e_1 v_1 e_2 \dots v_{l-1} e_l v_l$ in Φ has gain

$$\varphi(C) = \varphi(e_1; v_0, v_1)\varphi(e_2; v_1, v_2) \dots \varphi(e_l; v_{l-1}, v_l).$$

If C is a circle, we call it *balanced* when its gain is 1, the group identity. This property is independent of the starting point and direction. The *balanced circle class*

$$\mathcal{B}(\Phi) = \{C \in \mathcal{C}(\|\Phi\|) : C \text{ is balanced}\}$$

is fundamental. It is a *linear class* of circles: whenever two circles in a theta subgraph of Φ belong to $\mathcal{B}(\Phi)$, so does the third. A pair $(\Gamma, \mathcal{B})$ where $\mathcal{B} \subseteq \mathcal{C}(\Gamma)$ is a linear class is a *biased graph* [17]. The matroid and lattice theory of gain graphs (Sections 2–4) carry over to biased graphs, not always with easy proofs; but we de-emphasize biased graphs here since we do not need such generality.

An *unbalanced figure* is an unbalanced circle or half edge. An edge set or subgraph of Φ is *balanced* if it contains no unbalanced figure.

Such notations as $\Phi{:}X$, $\pi(\Phi)$, $c(\Phi)$ are as for graphs. Peculiar to Φ are the unbalanced vertex set of a subgraph Δ,

$$V_0(\Delta) = \bigcup \{B \in \pi(\Delta) : E(\Delta{:}B) \text{ is unbalanced}\},$$

and $V_0(S) = V_0(V, S)$ for $S \subseteq E$, the balanced partition

$$\pi_{\mathrm{b}}(\Delta) = \{B \in \pi(\Delta) : E(\Delta){:}B \text{ is balanced}\},$$

and the balanced component number

$$b(\Delta) = |\pi_{\mathrm{b}}(\Delta)|.$$

In particular we have $V_0(\Phi)$, $b(\Phi)$, etc. The *restriction* $\Phi|S$ of Φ to an edge set S is simply the spanning subgraph (V, S) with gains $\varphi\big|_S$. The *contraction* Φ/S is constructed in three steps. First we discard $V_0(S)$; any edge with one end in $V_0(S)$ and the other in the complement, at a vertex x say, becomes a half edge, while an edge whose endpoints are all in $V_0(S)$ becomes a loose edge. Then we switch so each balanced component of S has gain identically 1. Finally, we contract out the edges of S. (If S is closed in the bias matroid—see Section 2—no loose edges arise.) We see that $V(\Phi/S) = \pi_{\mathrm{b}}(S)$ and $E(\Phi/S)$ corresponds to $E \setminus S$ with endpoints $\omega_{\Phi/S}(e) = B(\omega_\Phi(e))$, where $B(v)$ is the block of $\pi_{\mathrm{b}}(S)$ to which $v \in V$ belongs (undefined if $v \notin \bigcup \pi_{\mathrm{b}}(S)$). Now we can define contraction of a (gainless) graph Γ by an arbitrary edge set S: one treats Γ as having gains $\varphi = 1$.

Some types of gain graph: a *full* gain (or biased) graph has an unbalanced edge (a half edge or unbalanced loop) at every vertex. If Φ is a gain graph, $\Phi^\bullet$ is Φ with a half edge added to each vertex not already supporting one. A gainless graph can for some purposes be treated as a gain graph with the trivial gain group $\{1\}$ [14, Section 7A]; then it is balanced precisely when it has no half edges. Let Γ be an ordinary graph: its $\mathfrak{G}$-*expansion* $\mathfrak{G}\Gamma$ is the gain graph having gain group $\mathfrak{G}$ and:

$$V(\mathfrak{G}\Gamma) = V(\Gamma);$$
$$E(\mathfrak{G}\Gamma) = \{(ge; v, w) : g \in \mathfrak{G},\ e \in E(\Gamma),\ \omega(e) = \{v, w\}\}/R$$

where R is the equivalance relation generated by $(ge; v, w)R(g^{-1}e; w, v)$;

$$\omega(ge; v, w) = \{v, w\};$$
$$\varphi(ge; v, w) = g.$$

The *full* $\mathfrak{G}$*-expansion* $\mathfrak{G}\Gamma^\bullet$ is $\mathfrak{G}\Gamma$ with a half edge at every vertex.

2. Structure theory for matroids on gain graphs

In this section we let $\Phi = (V, E, \varphi)$ be a gain graph with gain group $\mathfrak{G}$.

Before describing the matroids and lattices associated with a gain graph we briefly discuss switching and balance. *Switching* a gain graph Φ means replacing φ by the switched gain φ^λ, where $\lambda : V \to \mathfrak{G}$ is any vertex map (called a *switching function*) and

$$\varphi^\lambda(e; v, w) = \lambda(v)^{-1}\varphi(e; v, w)\lambda(w).$$

The switched graph Φ^λ is called *switching equivalent* to Φ. Clearly, switching leaves the gain of any circle unchanged up to conjugation; thus it does not affect balance of circles.

A *potential* for Φ is a mapping $f : V \to \mathfrak{G}$ such that

$$\varphi(e; v, w) = f(v)^{-1} f(w).$$

The natural multiplicative action of $\mathfrak{G}$ on a potential is on the left and on a switching function is on the right, as the first proposition implies.

Proposition 2.1. *Let Φ be a gain graph without half edges. The following are equivalent:*

(i) *Φ is balanced.*
(ii) *Φ has a potential function.*
(iii) *There is a switching function λ such that $\varphi^\lambda \equiv 1$.*

Proof. This is easy and well known. Note that $\lambda = f^{-1}$. ■

Now we come to the matroid theory. We treat the bias matroid, then the complete lift matroid.

The *balance-closure* of S is

$$\operatorname{bcl}(S) = S \cup \{e \in E \setminus S : e \text{ lies in a balanced circle in } S \cup \{e\}\}.$$

We say S is *balance-closed* if it is its own balance-closure. The following theorem characterizes the *bias matroid* of Φ. The cryptomorphisms are selected from eleven given in [17] and are lettered as in the source.

Theorem 2.2A ([17, Theorem II.2.1])**.** *Let Φ be a gain (or biased) graph with underlying graph (V, E). There is a matroid $G(\Phi)$ on E defined by any one of the following equivalent properties. The matroid is finitary. Let S denote any subset of E.*

(a) *The closure of S is* $\operatorname{clos}(S) = \operatorname{bcl}(S) \cup [E{:}V_0(S)]$. *(Clearly, it is balanced if S is.)*
(b) *S is closed (a flat) $\iff$ it equals $T \cup [E{:}X^c]$ where (X, T) is a balanced and balance-closed subgraph.*
(j) *The rank of S is*

$$\operatorname{rk}(S) = |V_0(S)| + \sum_{B \in \pi_b(S)} (|B| - 1)$$

and if $n = |V|$ is finite it is

$$\operatorname{rk}(S) = n - b(S).$$

(k) *The corank of S is $b(\Phi|S) - b(\Phi)$.* ■

Supposing Φ is balanced, $G(\Phi)$ is the polygon matroid $G(\|\Phi\|)$.

Theorem 2.3A ([Theorem II.2.5 and Lemma I.4.3]BG). *We have $G(\Phi|S) = G(\Phi)|S$ and $G(\Phi/S) = G(\Phi)/S$. If S is balanced and $A \supseteq S$, then A is balanced in $\Phi \iff A \setminus S$ is balanced in Φ/S.* ■

We shall resume the treatment of $G(\Phi)$ with Corollary 2, on the lattice theory, but first we give the analogs for lifts of the foregoing theorems.

The point set of the *complete lift matroid* $L_0(\Phi)$ is $E \cup \{e_0\}$ where e_0, the *extra point*, is not in Φ. The *lift matroid* $L(\Phi)$ is $L_0(\Phi) \setminus e_0$. We call a point set $S \subseteq E \cup \{e_0\}$ *balanced* if S is a balanced edge set (thus $e_0 \notin S$).

Theorem 2.2B ([17, Theorem II.3.1]). *Let Φ be a gain (or biased) graph with underlying graph (V, E). There is a matroid $L_0(\Phi)$ on the point set $E \cup \{e_0\}$, where e_0 is an extra point, given by the following equivalent properties. The matroid is finitary. Let S denote any subset of $E \cup \{e_0\}$.*

(a) *The closure of S is*

$$\operatorname{clos}(S) = \begin{cases} \operatorname{bcl}(S) & \text{if } S \text{ is balanced,} \\ \\ \operatorname{clos}_{\|\Phi\|}(S \setminus e_0) \cup \{e_0\} & \text{otherwise.} \end{cases}$$

(It is obviously balanced if S is.)

(b) *S is closed $\iff$ it is balanced and balance-closed or else it contains e_0 and $S \setminus e_0$ is polygon-closed in $\|\Phi\|$.*

(j) *The rank of S is*

$$\operatorname{rk}_{L_0}(S) = \sum_{B \in \pi(S)} (|B| - 1) + \varepsilon = \operatorname{rk}_{\|\Phi\|}(S) + \varepsilon,$$

where $\varepsilon = 0$ if S is balanced, 1 otherwise. If $n = |V|$ is finite

$$\operatorname{rk}_{L_0}(S) = \begin{cases} n - c(S) & \text{if } S \text{ is balanced,} \\ \\ n + 1 - c(S \setminus e_0) & \text{otherwise.} \end{cases}$$

(k) *The corank of S is $c(S)$ if S is balanced, $c(S \setminus e_0) - 1$ otherwise.*

Theorem 2.3B ([17, Theorem II.3.6]). *We have $L_0(\Phi)|S = L_0(\Phi|S)$ and*

$$L_0(\Phi)/S = \begin{cases} L_0(\Phi/S) & \text{if } S \text{ is balanced,} \\ \\ G(\|\Phi\|/(S \setminus e_0)) & \text{if } e_0 \in S, \\ \\ G(\|\Phi\|/S) + (e_0)_0 & \text{if } e_0 \notin S \text{ and } S \text{ is unbalanced.} \end{cases}$$

(Here $(e_0)_0$ denotes e_0 as a matroid loop.) ■

We define

$$\operatorname{Lat^b} \Phi = \{S \subseteq E : S \text{ is balanced and closed}\},$$

which is a meet semilattice but not in general a lattice, and the lattices

$$\operatorname{Lat} G(\Phi) = \{S \subseteq E : S \text{ is closed in } G(\Phi)\},$$
$$\operatorname{Lat} L_0(\Phi) = \{S \subseteq E \cup \{e_0\} : S \text{ is closed in } L_0(\Phi)\},$$

which of course both contain $\operatorname{Lat}^{\mathrm{b}} \Phi$ as a lower ideal. These three are the fundamental objects for our study. For the rest of this section we examine them from several sides.

First of all is the *edge form* of the three semilattices, which is that given by Theorems 2(b) and 2(b) and the above definitions. The partial order relation is that of containment; meet is set intersection; and the join of flats S_1 and S_2 is $\operatorname{clos}(S_1 \cup S_2)$, which may differ in G and L_0 even when S_1 and S_2 are balanced flats, hence common to both. We conclude our treatment of edge form with a few observations. Recall that

$$\operatorname{Lat} \Gamma = \{S \subseteq E(\Gamma) : S \text{ is closed in } G(\Gamma)\} \cong \Pi(\Gamma).$$

Corollary 2.2C ([17, Corollary II.4.1])**.** *Considering the lattices in edge form we have*

$$\operatorname{Lat}^{\mathrm{b}} \Phi = \operatorname{Lat} G(\Phi) \cap \operatorname{Lat} L_0(\Phi),$$

$$\operatorname{Lat} L(\Phi) = \operatorname{Lat}^{\mathrm{b}}(\Phi) \cup \operatorname{Lat} G(\|\Phi\|), \quad \textit{and}$$

$$\operatorname{Lat} L_0(\Phi) \cong \operatorname{Lat}^{\mathrm{b}}(\Phi) \uplus \operatorname{Lat} G(\|\Phi\|),$$

the latter a disjoint union where $S \in \operatorname{Lat} G(\|\Phi\|)$ corresponds to $S \cup \{e_0\} \in \operatorname{Lat} L_0(\Phi)$.

The maximal elements of $\operatorname{Lat}^{\mathrm{b}} \Phi$ are all of rank $n - c(\Phi)$ and corank $c(\Phi) - b(\Phi)$.

Proof. This is all clear except possibly the last remark. A maximal balanced flat is the closure of a maximal forest, hence has $c(\Phi)$ components (counting isolated vertices), all balanced. ∎

Since $\operatorname{Lat}^{\mathrm{b}} \Phi = \{A \in \operatorname{Lat} L_0(\Phi) : e_0 \notin A\}$, it is a geometric semilattice [10].

We can see more clearly what is going on in the lattices by recasting them in *subgraph form* where we think of the elements as balanced subgraphs and, in $\operatorname{Lat} L_0(\Phi) \setminus \operatorname{Lat}^{\mathrm{b}} \Phi$, as polygon-closed edge sets of $\|\Phi\|$. Let us define a partial ordering on balanced subgraphs of Φ by $(X, S) \leq (Y, T)$ if $X \supseteq Y$ and $S \subseteq E{:}(X \setminus Y) \cup T$. We also order a spanning subgraph and a pure edge set by $(V, S) < T$ if $S \subseteq T$ (but T is never $\leq (X, S)$).

Proposition 2.4A. *Let Φ be a gain (or biased) graph with underlying graph (V, E).*

The set of balanced and balance-closed subgraphs $(X, T) \subseteq \|\Phi\|$ such that $(^*)$ $\Phi{:}X^c$ has no balanced components, ordered as above, is isomorphic to $\operatorname{Lat} G(\Phi)$. The isomorphism is $S \to (X, S{:}X)$ where $X = V_0(S)^c$ for an edge set $S \in \operatorname{Lat} G(\Phi)$, and oppositely $(X, T) \to T \cup [E{:}X^c]$ for a subgraph (X, T). The bottom and top elements are $\bar{K}_V = (V, \varnothing)$ and $\|\Phi\|[V_0(\Phi)^c]$. The rank function (when $n = |V|$ is finite) is $\operatorname{rk}(X, T) = n - c(X, T)$. The join is $(X_1, T_1) \vee (X_2, T_2) = \operatorname{bcl}(X, T)$ where (X, T) is the union of the balanced components of $(X_1 \cup X_2,\ T_1 \cup T_2)$ that lie entirely in $X_1 \cap X_2$. The meet $(X_1, T_1) \wedge (X_2, T_2)$ is the union of

$$(X_1 \setminus X_2, T_1[X_1 \setminus X_2]) \cup (X_2 \setminus X_1, T_2[X_2 \setminus X_1]) \cup (X_1 \cap X_2, T_1 \cap T_2)$$

and $(^*)$ any balanced components of $\Phi{:}(X_1 \cup X_2)^c$.

If Φ is full, the $(^*)$ conditions may be omitted and the top element is the null graph.

Under the isomorphisms, the spanning subgraphs (V, T) correspond to $\operatorname{Lat}^{\mathrm{b}} \Phi$.

Proof. This proposition follows directly by translating the properties of flats from edge to subgraph form. ∎

Proposition 2.4B. *Let Φ be a gain (or biased) graph with underlying graph (V, E).*

The set $\operatorname{Lat} L_0(\Phi)$ is order isomorphic to the (disjoint) union of $\{(V, T) \subseteq \|\Phi\| : T$ is balanced and balance-closed$\}$ and $\operatorname{Lat} G(\|\Phi\|)$. The isomorphism is given by $S \to (V, S)$ if S is a balanced flat and $S \to S \setminus e_0$ if it is an unbalanced flat, and in the reverse direction

$(V,T) \to T$ and $T \to T \cup \{e_0\}$ for a subgraph and a polygon-closed edge set, respectively. The bottom and top elements are $\bar{K}_V$ and E. The rank function (if $|V| = n$ is finite) is $\operatorname{rk}(V,T) = n - c(V,T)$ *and* $\operatorname{rk}(T) = n + 1 - c(V,T)$. *The join is given by*

$$(V,T_1) \vee (V,T_2) = \begin{cases} (V, \operatorname{bcl}(T_1 \cup T_2)) & \text{if balanced,} \\ \\ E{:}(\pi(T_1 \cup T_2)) & \text{if not,} \end{cases}$$

$$\begin{aligned} (V,T_1) \vee T_2 &= E{:}(\pi(T_1 \cup T_2)), \\ T_1 \vee T_2 &= E{:}(\pi(T_1 \cup T_2)). \end{aligned}$$

The meet is intersection for two subgraphs or two polygon-closed edge sets, and $(V,T_1) \wedge T_2 = (V, T_1 \cap T_2)$.

Proof. This is another translation exercise. ■

Proposition 2.5A. *If Φ is the disjoint union of Φ_i, $i \in I$, then*

$$\operatorname{Lat}^{\mathrm{b}} \Phi \cong \bigtimes_{i \in I} \operatorname{Lat}^{\mathrm{b}} \Phi_i,$$

$$\operatorname{Lat} G(\Phi) \cong \bigtimes_{i \in I} \operatorname{Lat} G(\Phi_i). \qquad \blacksquare$$

Proposition 2.5B. *If Φ is as above, then*

$$\operatorname{Lat} L_0(\Phi) \cong \big(\bigtimes_{i \in I} \operatorname{Lat}^{\mathrm{b}} \Phi_i\big) \cup \big(\bigtimes_{i \in I} \Pi(\|\Phi_i\|)\big). \qquad \blacksquare$$

The purpose of listing the subgraph properties in such detail is to make them available in connection with particular matroids to be discussed later. It is convenient sometimes to work with edge form and sometimes with subgraph form. We consider the lattice notations $\operatorname{Lat} G(\Phi)$ *et al.* to denote either form; the context will clarify which it is.

3. Lattices of group-valued partitions

We now switch to an essentially latticial point of view based on partitions. This is the heart of our development of structured partition lattices.

Let $\mathfrak{G}$ be a fixed group. A *homogeneous B-tuple* in $\mathfrak{G}$ is an orbit of mappings $B \to \mathfrak{G}$, on a set B, under the left multiplicative action of $\mathfrak{G}$. We denote by $[\mathfrak{G}^B]$ the set of homogeneous B-tuples. A *$\mathfrak{G}$-valued partition* γ, briefly just a *$\mathfrak{G}$-partition*, consists of a set partition $\pi(\gamma)$ each of whose blocks B has an associated homogeneous B-tuple γ_B, called a *constituent* of γ. The *support* of γ is $\operatorname{supp} \gamma = \operatorname{supp} \pi(\gamma) = \bigcup \pi(\gamma)$. Apparently Dowling [3] first introduced the concept of a $\mathfrak{G}$-partition.

Technically we define γ as the ordered pair $(\operatorname{supp} \gamma, \{\gamma_B : B \in \pi(\gamma)\})$, so that a $\mathfrak{G}$-partition is not a partition, not even if it partitions the empty set. For convenience we often write simply $\gamma = \{\gamma_B\}$. Some terminology and notation: Choosing a representative of a homogeneous B-tuple γ_B defines a partition

$$\pi(\gamma_B) = \{\gamma_B^{-1}(g) : g \in \operatorname{Im}(\gamma_B)\}$$

of B, which is independent of the representative. (We denote the representative function by the same symbol γ_B because no ambiguity can arise.) Each block of $\pi(\gamma_B)$ is a *subblock* of γ; the *subblock partition* of $\operatorname{supp}\gamma$ is $\pi_0(\gamma) = \bigcup_B \pi(\gamma_B)$. A homogeneous B-tuple is *trivial* if it is a constant function; that is, if its partition is trivial. A *partial* $\mathfrak{G}$*-partition* of a set V is a $\mathfrak{G}$-valued partition of a subset of V. A particular $\mathfrak{G}$-partition of V is the *total* one, which we write o_V; it is characterized by having underlying partition 0_V.

Let V be a fixed set of order n (possibly infinite or zero). The bases of all our further developments are $P_n(\mathfrak{G})$, the set of $\mathfrak{G}$-valued partitions of V, $P_n^\dagger(\mathfrak{G})$, that of partial $\mathfrak{G}$-partitions, and $P_n^\S(\mathfrak{G}) = P_n(\mathfrak{G}) \cup \Pi_n$ (which by definition is a disjoint union). Each set is partially ordered by the *refinement* ordering $\leq$. Partitions are ordered by the usual refinement ordering. Let γ, δ be $\mathfrak{G}$-partitions and π a partition. We say γ is *finer* than δ, $\gamma \leq \delta$, if $\operatorname{supp}\gamma \supseteq \operatorname{supp}\delta$ and, for each block B of γ, either $B \subseteq (\operatorname{supp}\gamma) \setminus (\operatorname{supp}\delta)$ or else there is a block D of δ containing B such that $\gamma_B = \delta_D\big|_B$ (the restricted homogeneous function). This entails $\pi(\gamma) \leq \pi(\delta)$. Also, γ is *finer* than π, $\gamma \leq \pi$, if $\pi(\gamma) \leq \pi$. We never have $\pi \leq \gamma$. (We do not actually need ordering between a $\mathfrak{G}$-partition and a partition of different sets.) The refinement ordering of partial $\mathfrak{G}$-partitions was first introduced by Dowling [3].

The *restriction* of γ to a set X is

$$\gamma|_X = \{\gamma_B\big|_{X\cap B} : B \in \pi(\gamma),\ X \cap B \neq \varnothing\}.$$

Now we can define meet and join operations. First the meet of γ_1 and γ_2. Let $X_i = \operatorname{supp}\gamma_i$. On each block of $\pi(\gamma_1) \wedge \pi(\gamma_2)$, of the form $B = B_1 \cap B_2$ where $B_i \in \pi(\gamma_i)$, define an equivalence relation by

$$v \sim w \iff \gamma_{1B_1}(v)^{-1}\gamma_{1B_1}(w) = \gamma_{2B_2}(v)^{-1}\gamma_{2B_2}(w).$$

The *intersection* $\mathfrak{G}$-partition $\gamma_1 \cap \gamma_2$ consists of the blocks C of this equivalence relation on $X_1 \cap X_2$ with constituents $(\gamma_1 \cap \gamma_2)_C = \gamma_{1B_1}\big|_C = \gamma_{2B_2}\big|_C$, where $C \subseteq B_1 \cap B_2$. The *meet* is

$$\gamma_1 \wedge \gamma_2 = \gamma_1|_{X_1 \setminus X_2} \cup \gamma_2|_{X_2\setminus X_1} \cup (\gamma_1 \cap \gamma_2).$$

The blocks of the join are those blocks $C \in \pi(\gamma_1) \wedge \pi(\gamma_2)$ on which there is a homogeneous C-tuple γ_C (obviously unique) such that $\gamma_C\big|_{B_i} = (\gamma_i)_{B_i}$ for all $B_i \in \pi(\gamma_i|_C)$, $i = 1$ and 2. The *join* is then

$$\gamma_1 \vee \gamma_2 = \{\gamma_C : \gamma_C \text{ exists}\}.$$

(A necessary and sufficient condition for the existence of γ_C is that $\gamma_1|_B = \gamma_2|_B$ for each block B of $\pi(\gamma_1|_C) \wedge \pi(\gamma_2|_C)$.) Dowling first defined these meet and join operations [3].

We also define meet and join for $P_n^\S(\mathfrak{G})$. The meet is as in $P_n(\mathfrak{G})$ for $\mathfrak{G}$-partitions and as in Π_n for ordinary ones. For a mixed pair we define the *meet*

$$\gamma \wedge \pi = \bigcup_{C\in\pi} \gamma|_C.$$

The *join* is as in Π_n for ordinary partitions. For $\mathfrak{G}$-partitions it is

$$\gamma_1 \vee^\S \gamma_2 = \begin{cases} \gamma_1 \vee \gamma_2 & \text{if that has support } V, \\ \pi(\gamma_1) \vee \pi(\gamma_2) & \text{otherwise.} \end{cases}$$

For a mixed pair it is

$$\gamma \vee^\S \pi = \pi(\gamma) \vee \pi.$$

For our general lattice theorems we need to define constraints on $\mathfrak{G}$-valued partitions. Suppose we have a set V, a group $\mathfrak{G}$, and a mapping

$$\mathbf{I} : V \times V \to \mathcal{P}(\mathfrak{G})$$

satisfying the antisymmetric law $\mathbf{I}_{wv} = \mathbf{I}_{vw}^{-1}$, where we write $\mathbf{I}_{vw} = \mathbf{I}(v, w)$. Define an undirected simple graph

$$\Delta_{\mathbf{I}} = (V,\ \{vw : v \neq w,\ \mathbf{I}_{vw} \neq \varnothing\}).$$

We are interested in $\mathfrak{G}$-valued partitions that are $\mathbf{I}$-connected, meaning that the gain graph

$$\Psi(\gamma_B) = \Psi_{\mathbf{I}}(\gamma_B) = (B,\ \{ge_{vw} : g = \gamma_B(v)^{-1}\gamma_B(w) \in \mathbf{I}_{vw}\}, \psi_{\mathbf{I}}(\gamma_B)),$$

whose gain function is $\psi_{\mathbf{I}}(\gamma_B)(ge_{vw}) = g$, is connected for every constituent of γ. (Thus γ_B acts as a potential and $\Psi(\gamma_B)$ is balanced. Notice that $\mathbf{I}_{vv}$ is immaterial.) Briefly we call γ an $\mathbf{I}$-*partition*, leaving $\mathfrak{G}$ implicit. We also define

$$\Psi(\gamma) = \bigcup\{\Psi(\gamma_B) : B \in \pi(\gamma)\}$$

and $\Delta(\gamma) = \|\Psi(\gamma)\|$, $\Delta(\gamma_B) = \|\Psi(\gamma_B)\|$.

Let $P(\mathfrak{G}, \mathbf{I})$ and $P^\dagger(\mathfrak{G}, \mathbf{I})$ be respectively the sets of spanning and partial $\mathbf{I}$-partitions of V and let $P^\S(\mathfrak{G}, \mathbf{I}) = P(\mathfrak{G}, \mathbf{I}) \cup \Pi(\Delta_{\mathbf{I}})$. We partially order these sets as subsets of $P_n^\dagger(\mathfrak{G})$ and $P_n^\S(\mathfrak{G})$. Their join operations are also as in those sets but the meets are different. In $P^\dagger(\mathfrak{G}, \mathbf{I})$ we define

$$\begin{aligned}\gamma_1 \wedge_{\mathbf{I}}\ \gamma_2 &= \{(\gamma_1 \wedge \gamma_2)_B\big|_C : B \in \pi(\gamma_1 \wedge \gamma_2),\ C \in \pi(\Psi((\gamma_1 \wedge \gamma_2)_B))\} \\ &= \gamma_1 \wedge \gamma_2 \wedge \pi(\Psi(\gamma_1 \wedge \gamma_2)).\end{aligned}$$

This is the minimal refinement of $\gamma_1 \wedge \gamma_2$ into $\mathbf{I}$-connected constituents. In $P_n^\S(\mathfrak{G}, \mathbf{I})$ we define $\gamma_1 \wedge_{\mathbf{I}}\ \gamma_2$ as above and

$$\begin{aligned}\gamma \wedge_{\mathbf{I}}\ \pi &= \{(\gamma \wedge \pi)_B\big|_C : B \in \pi(\gamma \wedge \pi),\ C \in \pi(\Psi((\gamma \wedge \pi)_B))\} \\ &= \gamma \wedge \pi(\Psi(\gamma \wedge \pi)),\end{aligned}$$

which is also the least refinement of $\gamma \wedge \pi$ that gives $\mathbf{I}$-connected constituents. For partitions we have

$$\pi_1 \wedge_{\mathbf{I}}\ \pi_2 = \pi(\Delta_{\mathbf{I}}[\pi_1 \wedge \pi_2]),$$

which is the same as in $\Pi(\Delta_{\mathbf{I}})$. Finally, for $X \subseteq V$ we define $\mathbf{I}|X$ to be the restriction of $\mathbf{I}$ to $X \times X$ and for a partial partition π of V we define $\mathbf{I}|\pi = \bigcup_{X \in \pi}(\mathbf{I}|X)$.

Theorem 3.1A. *Let V of order $n \geq 0$, $\mathfrak{G}$, and $\mathbf{I}$ be given.*

(I) *The set $P^\dagger(\mathfrak{G}, \mathbf{I})$ of $\mathbf{I}$-connected partial $\mathfrak{G}$-partitions of V is a finitary geometric lattice of rank n. Its bottom and top elements, rank function, and join operation are as in $P_n^\dagger(\mathfrak{G})$. Its meet operation is $\wedge_{\mathbf{I}}$ defined above.*

The set $P(\mathfrak{G}, \mathbf{I})$ of $\mathbf{I}$-connected $\mathfrak{G}$-partitions of V is a lower ideal whose maximal elements have corank $c(\Delta_{\mathbf{I}})$.

(II) *An interval $[\gamma, \delta]$ in $P^\dagger(\mathfrak{G}, \mathbf{I})$ is isomorphic to*

$$P^\dagger(\mathfrak{G}, \mathbf{I}^*|\pi(\gamma|_X)) \times \bigtimes_{D \in \pi(\delta)} \Pi(\Delta(\delta_D)/\pi(\gamma|_D)),$$

where $\mathbf{I}^* : \pi(\gamma) \times \pi(\gamma) \to \mathcal{P}(\mathfrak{G})$ *is defined by choosing an arbitrary representative function for each* $\gamma_B \in \gamma$ *and setting*

$$\mathbf{I}^*_{AB} = \bigcup_{v \in A, w \in B} \gamma_A(v)\mathbf{I}_{vw}\gamma_B(w)^{-1},$$

and $X = (\operatorname{supp}\gamma) \setminus (\operatorname{supp}\delta)$.

Why the choice of representatives does not affect the isomorphism type will become apparent with Lemma 3. See Corollary 3.

Proof. We begin with the key definition and lemma. Let $\Phi = \Phi(\mathbf{I})$ be the gain graph with vertex set V, edge set

$$E = \{ge_{vw} : v \neq w,\ g \in \mathbf{I}_{vw}\},$$

where we regard ge_{vw} and $g^{-1}e_{wv}$ as the same edge with endpoints v and w, and gains

$$\varphi(ge_{vw};\ v, w) = g.$$

The gain group is $\mathfrak{G}$. Thus $\Psi_{\mathbf{I}}(\gamma) \subseteq \Phi$ for each $\gamma \in P^\dagger(\mathfrak{G}, \mathbf{I})$. We define

$$E{:}\gamma = E(\Psi_{\mathbf{I}}(\gamma)).$$

If (B, T) is a balanced and connected subgraph of Φ, let $\gamma_B(T)$ be a potential function for $(B, T, \varphi\big|_T)$. If $F = (X, S)$ is any subgraph of Φ, the associated $\mathfrak{G}$-partition is

$$\gamma(F) = (X, \{\gamma_B(S{:}B) : B \in \pi_{\mathrm{b}}(F)\}).$$

It is clear that $\gamma(F) \in P^\dagger(\mathfrak{G}, \mathbf{I})$, and that any representative of the constituent γ_B is a potential function for $\Psi_{\mathbf{I}}(\gamma_B)$, which is therefore balanced.

Lemma 3.1A. *The sets* $P^\dagger(\mathfrak{G}, \mathbf{I})$ *and Lat* $G(\Phi(\mathbf{I})^\bullet)$ *are order isomorphic by the inverse pair of isomorphisms* $\gamma \to \|\Psi(\gamma)\|$ *and* $F \to \gamma(F)$*. Under these isomorphisms* $P(\mathfrak{G}, \mathbf{I})$ *corresponds with* $\operatorname{Lat}^{\mathrm{b}} \Phi(\mathbf{I})$*.*

An equivalent result for $P_n^\dagger(\mathfrak{G})$, that is, for $\mathbf{I} \equiv \mathfrak{G}$ and $\Phi = \Phi(\mathbf{I}) = \mathfrak{G}K_n$, is proved in [2].

Proof. We are treating with $\operatorname{Lat} G(\Phi^\bullet)$ in subgraph form.

We have observed that $\|\Psi(\gamma)\| = (X, T)$ where T is balanced, and its vertex partition is $\pi(\gamma)$. We want to show it is balance-closed in Φ in order to conclude it is in Lat $G(\Phi^\bullet)$. We prove something stronger: $\|\Psi(\gamma(X, S))\| = (X, T)$ where $T = \operatorname{bcl} S$, if (X, S) is balanced. Clearly $\pi(\Psi(\gamma(X, S))) = \pi(\gamma(X, S)) = \pi(X, S)$; so the vertex set remains the same and we may focus on the connected case. Since (any representative X-tuple of) γ_X is a potential for (X, S), $S \subseteq T$. Suppose $ge_{vw} \in E{:}X \setminus S$ and take a path P in S from v to w. By the definition of $\Psi(\gamma)$, $ge_{vw} \in T$ if and only if $g = \gamma_X(v)^{-1}\gamma_X(w)$, which (since here $\gamma = \gamma(X, S)$) is true if and only if the circle $P + ge_{vw}$ is balanced. So $T = \operatorname{bcl} S$.

It follows that $\|\Psi(\gamma)\| \in \operatorname{Lat} G(\Phi^\bullet)$ (and $\in \operatorname{Lat}^{\mathrm{b}} \Phi$ if and only if $\operatorname{supp}\gamma = V$) and that $\gamma(\|\Psi(\gamma)\|) = \gamma$ and $\|\Psi(\gamma(F))\| = F$ for $\gamma \in P^\dagger(\mathfrak{G}, \mathbf{I})$ and $F \in \operatorname{Lat} G(\Phi^\bullet)$. That the two mappings preserve order is clear. So we have the lemma. ■

Proof of Theorem 3 continued. (I) All this is immediate except for meet and join and the corank of a maximal balanced flat. For the last named see Corollary 2. We must now prove join and meet in $P^\dagger(\mathfrak{G}, \mathbf{I})$ are as described.

For the join, consider flats F_1 and F_2 in subgraph form. A component (B, S) of $F_1 \cup F_2$ is balanced precisely when its gains have a potential, thus precisely when there is a homogeneous B-tuple γ_B for which $S \subseteq E{:}\gamma_B$. If (B, S) is balanced, then $\Delta(\gamma_B)$ is connected and in addition $\gamma_B\big|_{B_i}$ is a potential for $F_i[B_i]$ for each $B_i \in \pi(F_i{:}B)$, $i = 1$ and 2. We conclude that γ_B is a constituent of $\gamma(F_1 \cup F_2)$. But if (B, S) is unbalanced, B is disjoint from $\operatorname{supp}\gamma(F_1 \cup F_2)$ and also there is no γ_B which is compatible both with γ_{B_1} and γ_{B_2} for all $B_1, B_2 \subseteq B$. Since $F_1 \vee F_2 = \operatorname{bcl}(F)$ where F is the balanced part of $F_1 \cup F_2$ (the union of its balanced components; that is, $(F_1 \cup F_2)[\pi_{\mathrm{b}}(F_1 \cup F_2)]$), we conclude that $\gamma(F_1 \vee F_2) = \gamma(F_1 \cup F_2) = \gamma(F_1) \vee \gamma(F_2)$ in the definition we gave for join of $\mathfrak{G}$-partitions. From this and order isomorphism it follows that $\|\Psi(\gamma_1)\| \vee \|\Psi(\gamma_2)\| = \|\Psi(\gamma_1 \vee \gamma_2)\|$ for $\gamma_1, \gamma_2 \in P^{\dagger}(\mathfrak{G}, \mathbf{I})$.

For the meet consider flats $F_1 = (X_1, S_1)$ and $F_2 = (X_2, S_2)$ of $G(\Phi^{\bullet})$, with γ_1 and γ_2 their $\mathfrak{G}$-partitions. First we examine their intersection on a nonempty set $B = B_1 \cap B_2$, where $B_i \in \pi(F_i)$. We have

$$S_i{:}B = \{ge_{vw} \in E : v, w \in B \text{ and } g = \gamma_{iB_i}(v)^{-1}\gamma_{iB_i}(w)\},$$

whence

$$(S_1 \cap S_2){:}B = \{ge_{vw} \in E : v, w \in B \text{ and } \\ g = \gamma_{1B_1}(v)^{-1}\gamma_{1B_1}(w) = \gamma_{2B_2}(v)^{-1}\gamma_{2B_2}(w)\}.$$

The connected components of $(S_1 \cap S_2){:}B$ are therefore obtained by subdividing the equivalence classes C of the relation $\sim$ on B into the vertex sets of the connected components of $(S_1 \cap S_2){:}C = E{:}(\gamma_1 \wedge \gamma_2)|_C$. Therefore, on $X_1 \cap X_2$ both $S_1 \cap S_2$ and $\gamma_1 \wedge \gamma_2 \wedge \pi(\Psi(\gamma_1 \wedge \gamma_2))$ determine the same partition of the vertices and the same homogeneous functions on its blocks. Next we examine $F_1 \wedge F_2$ on $X_1 \setminus X_2$. Here it is simply $(X_1 \setminus X_2, S_1[X_1 \setminus X_2])$, whose $\mathfrak{G}$-valued partition is $\gamma_1|_{X_1 \setminus X_2} \wedge \pi(\Psi(\gamma_1|_{X_1 \setminus X_2}))$. Thus we have the description of meet given earlier for $P^{\dagger}(\mathfrak{G}, \mathbf{I})$.

(II) For this part it is best to take flats of $G(\Phi^{\bullet})$ in edge form. Let F and H be flats with $F \subseteq H$. We need $(G(\Phi^{\bullet})|H)/F$, which is $G(H/F)$ by Theorem 2. Let $\gamma = \gamma(F)$, $\delta = \gamma(H)$, and $X = (\operatorname{supp}\gamma) \setminus (\operatorname{supp}\delta) = V_0(H) \setminus V_0(F)$. Then H/F is the disjoint union of $(\Phi^{\bullet}{:}X)/(F{:}X)$ and all the contractions $H[D]/F[D]$ for $D \in \pi_{\mathrm{b}}(V, H) = \pi(\delta)$. But the latter are the same as $\|\Psi(\delta|_D)\|/\pi(\gamma|_D)$ since $H[D]$ is balanced and since our convention is to ignore loops formed by collapsing a partition. Because $\|\Psi(\delta|_D)\| = \Delta(\delta|_D)$, the portion of Lat $G(H/F)$ arising from the balanced components of H is the product factor in the statement of part (II).

The factor arising from the unbalanced part of H is the lattice of $G((\Phi^{\bullet}{:}X)/(F{:}X))$. The contraction graph has vertex set $\pi(X, F{:}X) = \pi(\gamma|_X)$. We have to switch from the gains φ of $\Phi^{\bullet}{:}X$ to φ^h in which $F{:}B$ has identity gains for all $B \in \pi(\gamma|_X)$. A switching function is $h(v) = \gamma_B(v_B)^{-1}\gamma_B(v)$ for $v \in B$ if we choose an arbitrary basepoint v_B in each B. Contracting by $F{:}X$ gives us edges between B and $C \in \pi(\gamma|_X)$ having all the gains $g \in h(v)\mathbf{I}_{vw}h(w)^{-1}$ for any $v \in B$ and $w \in C$, and no others. Switching the gains again by $f(B) = \gamma_B(v_B)^{-1}$, we see that $\mathbf{I}^*|\pi(\gamma|_X)$ describes the gains in $(\Phi{:}X)/(F{:}X)$, and we have part (II) from Lemma 3. ■

Theorem 3.1B. *Let V of order $n \geq 0$, $\mathfrak{G}$, and $\mathbf{I}$ be given.*

(I) *The set $P^{\S}(\mathfrak{G}, \mathbf{I})$ is a finitary geometric lattice of rank $n + 1 - c(\Delta_{\mathbf{I}})$. The bottom element, rank function, and join are as in $P^{\S}_n(\mathfrak{G})$. The top element is $\pi(\Delta_{\mathbf{I}})$ and meet is $\wedge_{\mathbf{I}}$ defined above.*

(II) *An interval* $[\gamma,\pi]$ *in* $P^{\S}(\mathfrak{G},\mathbf{I})$ *is isomorphic to*

$$\left(\bigtimes_{C\in\pi} P(\mathfrak{G},\mathbf{I}^*|\pi(\gamma|_C))\right)\cup\left(\bigtimes_{C\in\pi}\Pi\left(\Delta_{\mathbf{I}^*|\pi(\gamma|_C)}\right)\right),$$

where $\mathbf{I}^*$ *is as in Theorem 3.*

It may be desirable to clarify the interpretation of the union in part (II). The left product can be regarded in a natural way as a set of $\mathfrak{G}$-partitions of $\pi(\gamma)$ and the right product as a set of partitions of $\pi(\gamma)$. The union is therefore naturally isomorphic to $P^{\S}(\mathfrak{G},\mathbf{I}^*|\pi)$, where $(\mathbf{I}^*|\pi)_{B_1B_2}$ is defined to be $\mathbf{I}^*_{B_1B_2}$ or $\varnothing$ according as B_1 and B_2, blocks of γ, do or do not lie in a common block of π.

Proof. We need new maps involving partitions. Let $E_0(\gamma)=E{:}\gamma$ and let $\gamma(S)=\gamma(V,S)$ for a balanced edge set S, and let

$$E_0(\pi)=[E{:}\pi]\cup\{e_0\},\ \gamma(S\cup\{e_0\})=\pi(V,S)$$

for a partition $\pi\in\Pi_n$ and $S\subseteq E$.

Lemma 3.1B. *The mappings* $E_0(\,\cdot\,)$ *and* $\gamma(\,\cdot\,)$ *are inverse order isomorphisms of* $P^{\S}(\mathfrak{G},\mathbf{I})$ *with* Lat $L_0(\Phi)$*, under which* $P(\mathfrak{G},\mathbf{I})$ *corresponds to*$\mathrm{Lat}^{\mathrm{b}}(\Phi)$.

Proof. For pairs of partitions this is a well known fact about the polygon matroid. For $\mathfrak{G}$-partitions, or balanced flats, it is in Lemma 3. For a pair consisting of a $\mathfrak{G}$-partition γ whose flat is (X,S) and a partition π whose flat is $T\cup\{e_0\}$, we see that $S\subseteq T$ if and only if $E{:}\pi(V,S)\subseteq T$, hence if and only if $E_0(\pi(\gamma))\subseteq T\cup\{e_0\}=E_0(\pi)$. In other words, $S\subseteq T$ precisely when $\pi(\gamma)\le\pi$, as we wished to prove. ■

Proof of Theorem 3 continued. (I) The only parts that are not obvious are those concerning the join and meet operations in $P^{\S}(\mathfrak{G},\mathbf{I})$; we must show they are $\vee^{\S}$ and $\wedge_{\mathbf{I}}$ defined above.

Let γ_1 and γ_2 in $P(\mathfrak{G},\mathbf{I})$ correspond to flats $S_i=E{:}\xi_i$. If $S_1\cup S_2$ is balanced, then its closure is $\mathrm{bcl}(S_1\cup S_2)$, which is the balanced flat corresponding to $\gamma_1\vee^{\S}\gamma_2$. If on the other hand $S_1\cup S_2$ is unbalanced, then $\mathrm{supp}(\gamma_1\vee^{\S}\gamma_2)\neq V$; and the closure of $S_1\cup S_2$ is the flat $E_0[\pi(V,S_1)\vee\pi(V,S_2)]$. This settles the join of γ_1 and γ_2. A similar argument shows the join of γ and π is $\pi(\gamma)\vee\pi$.

The meet of γ_1 and γ_2 corresponds to $S_1\cap S_2$, hence is as in $P^{\dagger}(\mathfrak{G},\mathbf{I})$. That of γ and π is the $\mathfrak{G}$-valued partition corresponding to the balanced edge set

$$[E{:}\gamma]\cap[E{:}\pi]=\bigcup_{C\in\pi}E{:}\gamma|_C=E{:}(\gamma\wedge\pi)$$

by the definition of $\gamma\wedge\pi$. Since $(V,E{:}(\gamma\wedge\pi))=\Delta(\gamma\wedge\pi)$, we conclude that γ and π have meet $\gamma(\Delta[\gamma\wedge\pi])$. Clearly this is obtained by breaking up the blocks of $\gamma\wedge\pi$ into $\Delta(\gamma\wedge\pi)$-connected components; this is $\gamma\wedge\pi\wedge\pi(\Delta(\gamma\wedge\pi))$. Since $\pi(\Delta(\gamma\wedge\pi))\le\pi$, we have proved the meet is as claimed.

(II) As in part (II) of the preceding theorem, by Theorem 2 we must calculate the contraction $[\Phi{:}\pi]/[E{:}\gamma]$. Each component is, as in the proof of Theorem 3(II), the graph Φ corresponding to $\mathbf{I}^*|C$. Then we have

$$[\gamma,\pi]\cong\mathrm{Lat}\,L_0([\Phi{:}\pi]/[E{:}\gamma])\cong P^{\S}(\mathfrak{G},\mathbf{I}^*_\pi),$$

where $(\mathbf{I}^*_\pi)_{AB}=\mathbf{I}^*_{AB}$ if A and B are in the same block of π, and $=\varnothing$ otherwise. ■

Corollary 3.1A. *Let V, $\mathfrak{G}$ and $\mathbf{I}$ be given. Let h be any monomorphism of $\mathfrak{G}$ into itself and let $f : V \to \mathfrak{G}$ be arbitrary. Define $\mathbf{I}'$ by $\mathbf{I}'_{vw} = f(v)^{-1}h(\mathbf{I}_{vw})f(w)$. Then $P^\dagger(\mathfrak{G}, \mathbf{I}')$ and $P^\S(\mathfrak{G}, \mathbf{I}')$ are isomorphic to $P^\dagger(\mathfrak{G}, \mathbf{I})$ and $P^\S(\mathfrak{G}, \mathbf{I})$ by natural isomorphisms agreeing on $P(\mathfrak{G}, \mathbf{I}')$, which is carried to $P(\mathfrak{G}, \mathbf{I})$.*

Proof. The isomorphisms are those corresponding to altering $\Phi^\bullet$ by applying h to all gains and then switching by f. ■

Let us denote by

$$i_{\mathbf{I}} : P^\dagger(\mathfrak{G}, \mathbf{I}) \to P^\dagger_n(\mathfrak{G}) \quad \text{and} \quad i_{\mathbf{I}} : P^\S(\mathfrak{G}, \mathbf{I}) \to P^\S_n(\mathfrak{G})$$

the natural injections. By Theorems 3, 3 they are join-preserving. Hence they have *adjoint* maps

$$i^\Delta_{\mathbf{I}} : P^\dagger_n(\mathfrak{G}) \to P^\dagger(\mathfrak{G}, \mathbf{I}) \quad \text{and} \quad i^\Delta_{\mathbf{I}} : P^\S_n(\mathfrak{G}) \to P^\S(\mathfrak{G}, \mathbf{I}),$$

given by

$$\begin{aligned} i^\Delta_{\mathbf{I}}(\gamma) &= \sup\{\delta \in P^\dagger(\mathfrak{G}, \mathbf{I}) : \delta \le \gamma\}, \\ i^\Delta_{\mathbf{I}}(\pi) &= \sup\{\rho \in P^\S(\mathfrak{G}, \mathbf{I}) : \rho \le \pi\}. \end{aligned} \tag{3.1}$$

The adjoints satisfy $i_{\mathbf{I}} i^\Delta_{\mathbf{I}} =$ identity. Note that the two $i^\Delta_{\mathbf{I}}$ agree on $P_n(\mathfrak{G})$ and that $i^\Delta_{\mathbf{I}}(\pi)$ is an ordinary partition.

4. Chromatic invariants

In this section we show how to calculate invariants of group-valued partition lattices through counting colorings of the associated gain graph. Invariants determinable in this way we call *chromatic*; they include the characteristic polynomial and the Whitney numbers. We assume throughout that V and $\mathfrak{G}$ are finite of orders n and M, except where we indicate otherwise.

A *k-coloring* of Φ, whose gain group is $\mathfrak{G}$, is a mapping $\theta : V \to ([k] \times \mathfrak{G}) \cup \{0\}$; it is *zero-free* if it does not assume the value 0. Its *improper edge set* $I(\theta)$ consists of every edge that is either a half edge or a loop at a vertex colored zero, or a balanced loop anywhere, or a loose edge, or a link e between v and w such that $\theta(v) = (l, g)$ and $\theta(w) = (l, h)$ with $h = g\varphi(e; v, w)$ or $\theta(v) = \theta(w) = 0$. If $I(\theta) = \varnothing$, θ is *proper*. The *zero-free* or *balanced chromatic polynomial* $\chi^*_\Phi(y)$ is defined by $\chi^*_\Phi(kM) =$ the number of proper zero-free k-colorings of Φ; the *chromatic polynomial* $\chi_\Phi(y)$ is defined by $\chi_\Phi(kM+1) =$ the number of proper k-colorings of Φ. The *zero-free* (or *balanced) Whitney number polynomial* $w^*_\Phi(x, y)$ is defined by

$$w^*_\Phi(x, kM) = \sum_{\kappa\lambda} x^{I(\kappa\lambda)},$$

summed over all zero-free k-colorings $\theta = (\kappa, \lambda) = \kappa\lambda$, and the *Whitney number polynomial* $w_\Phi(x, y)$ is defined by

$$w_\Phi(x, kM+1) = \sum_\theta x^{|I(\theta)|},$$

summed over all k-colorings. The crucial result for our purposes is:

Proposition 4.1. *We have*

$$w^*_\Phi(x, y) = y^{c(\Phi)} w_{\operatorname{Lat^b} \Phi}(x, y) \ \textit{and} \ w_\Phi(x, y) = y^{b(\Phi)} w_{\operatorname{Lat} G(\Phi)}(x, y).$$

Furthermore,

$$\chi^*_\Phi(y) = y^{c(\Phi)} p_{\mathrm{Lat^b}\,\Phi}(y) \text{ and } \chi_\Phi(y) = y^{b(\Phi)} p_{\mathrm{Lat}\,G(\Phi)}(y)$$

if Φ has no balanced loops or loose edges, both chromatic polynomials being identically zero otherwise.

Proof. Immediate from [17, Sections III.4 and III.5]. ■

By applying Proposition 4.1 in reverse we can define chromatic and Whitney number polynomials of a finite gain graph even when the gain group is infinite. This is essentially the approach in [17, Part III].

We call the *balanced*, *bias*, and *complete lift Whitney numbers of* Φ the Whitney numbers of *Latb*Φ, $\mathrm{Lat}\,G(\Phi)$, and $\mathrm{Lat}\,L_0(\Phi)$, except that $w_i(\Phi) \equiv 0$ if Φ has a loose edge or balanced loop. Proposition 4.1 gives us the means of calculating the first two through the corresponding coloring polynomials. In general it is much easier to evaluate the balanced polynomials, and fortunately this suffices to get the others.

Theorem 4.2A ([17, Theorem III.6.1])**.** *Let Φ, but not necessarily $\mathfrak{G}$, be finite. We have the expansion formulas*

$$\chi_\Phi(y) = \sum_{\substack{X\subseteq V \\ X^c \text{ stable}}} \chi^*_{\Phi:X}(y-1),$$

$$w_\Phi(x,y) = \sum_{X\subseteq V} x^{\mathrm{rk}\,\Phi:X^c} w^*_{\Phi:X}(x, y-1),$$

provided there are no loose edges in Φ. ■

The main proof in [17] is algebraic. In [17] and [15] we gave a combinatorial proof (for the two-element group, in [15] which is valid for any finite gain group. Here we describe a combinatorial technique by which we can prove Theorem 4 for the infinite gain groups of interest to us. Let $\mathfrak{G}$ be abelian but not finite. We can find an M so large that setting $g^M = 1$ for all g does not change the balance of any circle in Φ. ($\mathfrak{G}$ should be finitely generated; to assure this we take the subgroup generated by the gain values.) In the new group Theorem 4 holds, and also Proposition 4.1 since balance and $G(\Phi)$ remain the same. Therefore, we have proved the theorem for infinite abelian groups by a combinatorial argument.

For the complete lift we have also a reduction formula, one that is valid (*mutatis mutandis*) for any matroid complete lift.

Theorem 4.2B ([17, Theorem III.5.2])**.** *We have (without assuming $\mathfrak{G}$ is finite)*

$$w_{L_0(\Phi)}(x,y) = y^{-c(\Phi)}[(y-1)w^*_\Phi(x,y) + x w_\Gamma(x,y)]$$

and

$$p_{L_0(\Phi)}(y) = y^{-c(\Phi)}(y-1)\chi^*_\Phi(y). \quad ■$$

We have reduced the evaluation problem for all polynomials, and therefore for the Whitney numbers and the Möbius invariants, to the problem of calculating two balanced polynomials – or even one, since obtaining $w(x,y)$ gives $p(y)$. Although we have treated only chromatic invariants (because they are especially meaningful from the viewpoint of lattices), there are analogous results for dichromatic invariants, in particular the Tutte polynomial; see [17, Part III].

We conclude with three new reduction formulas for the balanced polynomials. The first applies when the gain graph is obtained by expanding one over a smaller group. Suppose Φ is a gain graph with group $\mathfrak{G}$. Any homomorphism $\rho : \mathfrak{G} \to \mathfrak{H}$ makes Φ into a new gain graph $\rho\Phi$ over $\mathfrak{H}$ with gains $\rho\varphi$. (Because our interest is primarily in lattices, in $\rho\Phi$ we may prune balanced loops and also parallel edges with the same gain.) In particular let us map $\mathfrak{G}K_n$ to $\mathfrak{H}K_n$. If Ψ is a subgraph of the latter, let $\rho^{-1}\Psi$ be its inverse image in the former. Thus $\rho^{-1}\Psi$ has the same vertices as Ψ and, for each edge $f = he_{vw} \in E(\Psi)$ with gain $\varphi(f; v, w) = h$, $\rho^{-1}\Psi$ has all edges ge_{vw} with gains $g \in \rho^{-1}(h)$.

Proposition 4.2. *Let Ψ be a finite gain graph of order n over the gain group $\mathfrak{H}$ and let $\rho : \mathfrak{G} \to \mathfrak{H}$ be an epimorphism from a finite group $\mathfrak{G}$. Let $\Phi = \rho^{-1}\Psi$ and let K be the size of Ker ρ. Then*

$$w^*_\Phi(x, y) = K^n w^*_\Psi(x, \frac{y}{K})$$

and $\chi^*_\Phi(x, y) = K^n \chi^*_\Psi(x, \frac{y}{K})$.

Proof. We denote a zero-free coloring by $\kappa\lambda$ where κ maps into $[k]$ and λ into $\mathfrak{G}$ or $\mathfrak{H}$. Let $\mathbf{I}_{vw}(\Phi)$ be the set of gains of edges in Φ from v to w. Then $\mathbf{I}_{vw}(\Phi) = \rho^{-1}\mathbf{I}_{vw}(\Psi)$. We have, for a zero-free coloring $\kappa\lambda$ of Φ, the improper edge set

$$I_\Phi(\kappa\lambda) = \{ge_{vw} : \kappa(v) = \kappa(w) \text{ and } g = \lambda(v)^{-1}\lambda(w) \in \mathbf{I}_{vw}(\Phi)\}.$$

Any coloring of Ψ has the form $\kappa \cdot \rho\lambda$; its improper edge set is

$$\begin{aligned} I_\Psi(\kappa \cdot \rho\lambda) &= \{he_{vw} : \kappa(v) = \kappa(w) \text{ and } h = \rho\lambda(v)^{-1}\rho\lambda(w) \in I_{vw}(\Psi)\} \\ &= \rho I_\Phi(\kappa\lambda). \end{aligned}$$

Therefore, $\operatorname{rk} I_\Psi(\kappa \cdot \rho\lambda) = \operatorname{rk} I_\Phi(\kappa\lambda)$. Each coloring $\kappa\mu$ of Ψ arises from exactly K^n of Φ such that $\kappa\mu = \kappa \cdot \rho\lambda$; the improper edge set of each of them has the same rank. Therefore,

$$\begin{aligned} w^*_\Phi(x, k|\mathfrak{G}|) &= \sum_{\kappa,\mu} \sum_{\lambda:\rho\lambda=\mu} x^{\operatorname{rk} I_\Phi(\kappa\lambda)} \\ &= \sum_{\kappa,\mu} K^n x^{\operatorname{rk} I_\Psi(\kappa\lambda)} = K^n w^*_\Psi(x, k|\mathfrak{H}|). \end{aligned}$$

This is true for all k and thus is a polynomial identity.

The chromatic polynomial formula follows by setting $x = 0$, except for the trivial case where there exist balanced loops or loose edges. ■

The original and most important case is that in which $\mathfrak{H}$ is trivial. Then $w^*_\Psi = w_{\|\Phi\|}$, $\chi^*_\Psi = \chi_{\|\Phi\|}$, and $K = |\mathfrak{G}|$. The formulas can be found in [13, Example 8] and, with proof, in [17, Example II.3.6] and, for $|\mathfrak{G}| = 2$, in [16], Theorem 3.1].

The second reduction is useful when we can calculate the balanced Whitney number (or just chromatic) polynomial at $y = M$ but not for other values.

Proposition 4.3. *For a finite gain graph Φ without loose edges we have the reduction*

$$w^*_\Phi(x, y) = \sum_{\pi \in \Pi_n} \left(\frac{y}{M}\right)_{|\pi|} \prod_{B \in \pi} w^*_{\Phi:B}(x, M)$$

and a similar formula for $\chi^*_\Phi(y)$.

Proof. For an algebraic proof see [17, Theorem III.7.1]. We give a combinatorial proof that is valid when the gain group is finite. A zero-free k-coloring is obtained by partitioning V, say by π, assigning an integer color $\kappa(B)$ to each vertex in each block $B \in \pi$, where $\kappa : \pi \to [k]$ is injective, and 1-coloring each $\Phi{:}B$. The number of injections is $(k)_{|\pi|}$. The improper edge set of Φ-coloring is the vertex-disjoint union of those of the 1-colorings of blocks; hence its rank is the sum of their ranks. ■

Suppose Φ is uniform, in the sense that every induced subgraph of a given order is the same, let us say $\Phi{:}X \cong \Phi_m$ where $m = |X|$. Then the proposition becomes

$$w^*_\Phi(x, y) = \sum_{r=0}^{n} \left(\frac{y}{M}\right)_r \sum_{\substack{n_1+\ldots+n_r=n \\ n_1\geq\ldots\geq n_r>0}} \prod_{i=1}^{r} w^*(\Phi_{n_i}; x, M). \tag{4.1}$$

The third reduction formula involves varying the identity portion of the gain graph. By Φ/π we mean $\|\Phi\|/\pi$ with edges having the same gains as in Φ and with balanced loops removed.

Proposition 4.4. *Let Φ be a finite gain graph over a finite group, and let Γ_1, Γ_2 be ordinary graphs, all on the same vertex set. Assume that $\Gamma_1 \subseteq \Gamma_2$ and that Φ has no edges with gain 1. Let Φ_Γ denote $\Phi \cup \{1\}\Gamma$. Then*

$$\chi^*(\Phi_{\Gamma_1}; y) = \sum_{\substack{\pi\in\Pi(\Gamma_2) \\ E(\Gamma_1{:}\pi)=\varnothing}} \chi^*(\Phi_{\Gamma_2}/\pi; y).$$

Proof. Let $y = kM$. A zero-free proper k-coloring of Φ_{Γ_1}, say $\kappa\lambda$, may have improper edges in Φ_{Γ_2}, but only edges of gain equal to 1. Let $\pi = \pi(V, I_{\Gamma_2}(\kappa\lambda))$. Then $\kappa\lambda$ is constant on each block of π. It follows that $\Gamma_1{:}\pi$ has no edges and that $\kappa\lambda$ induces a proper k-coloring of Φ_{Γ_2}/π.

Conversely let $\kappa^*\lambda^*$ be a proper zero-free k-coloring of Φ_{Γ_2}/π for some $\pi \in \Pi(\Gamma_2)$ such that $E(\Gamma_1{:}\pi) = \varnothing$. We pull it back to a unique zero-free k-coloring $\kappa\lambda$ of Φ_{Γ_2} which is constant on blocks of π. The improper edge set I of $\kappa\lambda$ in Φ_{Γ_2} consists only of gain-1 edges within blocks of π, because $\kappa^*\lambda^*$ is proper, and it contains every such edge. Thus it equals $E(\Gamma_2{:}\pi)$, which has no Γ_1-edges by choice of π. Consequently, $\kappa\lambda$ is a proper coloring of Φ_{Γ_1}. Since $\pi \in \Pi(\Gamma_2)$, $\pi(V, I) = \pi$. Therefore, we have a pair of inverse one-to-one mappings between the sets of colorings counted on the left and right sides of the formula to be proved. That establishes the result. ■

Some interesting applications arise by taking $\Phi = [\mathfrak{G} \setminus \{1\}]\Gamma$. In case $\Gamma = \Gamma_2$ we deduce from Proposition 4.3 that

$$\chi^*(\Phi_{\Gamma_1}; y) = \sum_{\substack{\pi\in\Pi(\Gamma) \\ E(\Gamma_1{:}\pi)=\varnothing}} M^{|\pi|} \chi_{\Gamma/\pi}\left(\frac{y}{M}\right). \tag{4.2}$$

This yields the balanced chromatic polynomial in case $\Gamma_1 \subseteq \Gamma$. On the other hand, we can set $\Gamma_2 = K_n$ and then solve for any Γ_1 whatsoever by means of Theorem 5A(III) of Chapter II.

5. Comments and problems

Our work on particular structured partition lattices and associated matroids raises questions about the properties of abstractly structured partitions, the pairing of bias and lift matroids and their respective lattices, the possibility of more complex mixed-type lattices than the $P^{\S}$ lattices, and the orientation structures underlying the geometrical realizations of the real-representable matroids. The following remarks and suggestions on these topics are regrettably but necessarily vague since few of the ideas have been carried to completion.

5.1. **Abstraction.**

We are concerned in this series only with concrete examples. This leads us to wonder about the abstract nature and properties of structured partitions.

Problem 5.1. *Axiomatize structured partitions and their order properties.*

At minimum, a *structured partition* σ is a set partition $\pi(\sigma)$ each of whose blocks B carries a "structure" σ_B, whose exact nature we leave unspecified. To imitate the refinement/restriction partial ordering employed in our examples we need a notion of restriction: if σ_B is a structure on a set B, and if $S \subseteq B$, there is a set $R_S(\sigma_B)$ of structures on S called *restrictions* of σ_B. (Possibly $R_S(\sigma_B)$ is void; or it may even have several members.) Define $\sigma \leq \tau$ if $\pi(\sigma) \leq \pi(\tau)$ and, for each block $B \in \pi(\sigma)$ contained in a block $C \in \pi(\tau)$, $\sigma_B \in R_B(\tau_C)$. One can easily axiomatize the properties that make $\leq$ a partial ordering. But the resulting systems are far wilder than our examples. It seems likely that the above definitions do not capture many fundamental properties of structured partitions. They certainly do not guarantee the existence of meets or joins nor do they enable us to distinguish between types of structures, as we need to do in $P^{\S}$ lattices, where the two types, "structured" and "ordinary" partitions, are mixed.

Problem 5.2. *Axiomatize geometric lattices of structured partitions, in particular those with the standard rank function* $\operatorname{rk} \sigma = \operatorname{rk}_{\Pi_n} \pi(\sigma) = n - |\pi(\sigma)|$.

Certain geometric lattices of structured partial partitions of a set having the standard rank function $n - |\pi(\gamma)|$ are classifiable: they are equivalent to bias matroids of full biased graphs. (This result will be presented elsewhere.) It should be easy to extend the classification to lattices of structured and ordinary partitions of a set. There still remain probably many examples of mixed-type partitions that I do not even know how to define (cf. the commentary on Problem 5.5).

A general solution to Problem 5.2 may answer the difficult and vexing question of how to define biased hypergraphs and their matroids. This applied to signed hypergraphs, for instance, should have important consequences for $\{0, \pm 1\}$-optimization problems.

5.2. **Unification.**

Our entire theory is presented in metaphorical parallel columns: bias matroids and lattices of structured partial partitions on the left, lift matroids and lattices of structured and ordinary partitions on the right.

Problem 5.3. *Unify the two types of lattices.*

Any solution will cast light on Problems 5.1 and 5.2.

Notwithstanding possible general theories, it seems that G and L_0 are especially important. How are they related? Certainly $L_0(\Phi)$ is a weak image of $G(\Phi^\bullet)$, but we have no theoretical description of how this weak map operates, nor of what it entails for the lattices.

Problem 5.4. *Describe the matroid relationships between $G(\Phi^\bullet)$ and $L_0(\Phi)$, and the lattice relationships between $P^\dagger$ and $P^\S$.*

5.3. **Combination.** The lattice $P^\S$, constructed of structured and ordinary partitions, suggests that it may be possible to build geometric lattices combining structured partitions of several types.

Problem 5.5. *Construct geometric lattices combining partitions bearing "stronger" and "weaker" structures.*

The weakest structure being none at all, $P^\S$ is an example. I have in mind an approach through strength of bias. Let $\Phi_i = (\Gamma, \mathfrak{G}_i, \varphi_i)$ be gain graphs on the same base graph. We call Φ_i *stronger* than Φ_j if $\mathcal{B}(\Phi_i) \subset \mathcal{B}(\Phi_j)$. An example from Chapter III: composed partitions ($\mathfrak{G} = \mathbb{Z}$; $\operatorname{Im}\varphi \subseteq \{0, \pm 1\}$) are stronger than M-modularly composed partitions ($\mathfrak{G} = \mathbb{Z}_M$; same φ), which are stronger than m-modularly composed partitions if m is a proper divisor of M. Can we, therefore, build geometric lattices out of composed partitions and (say) bisected partitions (or partial partitions) of a set? Can we combine three or more types? If such constructions are indeed possible, we have a further open problem.

Problem 5.6. *Axiomatize strength relations among structure types. Axiomatize sets and lattices of structured partitions of mixed types.*

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