GEOMETRIC LATTICES OF STRUCTURED PARTITIONS
II. LATTICES OF GROUP-VALUED PARTITIONS BASED ON GRAPHS AND SETS

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Abstract. We use the methods of Chapter I, "Gain-graphic matroids and group-valued partitions", to treat several examples of geometric lattices based on a graph and a group. We give complete descriptions of meet, join, intervals (in most cases), Möbius invariant and (sometimes) Möbius function, characteristic polynomial, and simply and doubly indexed Whitney numbers. Half of our lattices are join subsemilattices of Dowling's lattices; the other half are similarly related to a new type of group-based geometric lattice related to the matroid lift construction.

Preface

In [1] Dowling introduced a construction that, for each $n \geq 0$ and each group $\mathfrak{G}$, gives a geometric lattice $P_n^{\dagger}(\mathfrak{G})$ of rank n. The elements of the lattice are the $\mathfrak{G}$-valued partial partitions of an n-element set. Dowling computed not only the lattice operations and interval structure of the lattice but also the characteristic polynomial and the Whitney numbers of the first and second kinds. Here we apply the method of gain graphs developed in Chapter I [4] to enlarge Dowling's results to cover new geometric lattices based, in different ways, on a group and a graph or a subset of the ground n-set. Our new lattices generalize lattices of special signed graphs (group of order 2; see [2, Section 7] and [3]). They are join subsemilattices of Dowling's; what is noteworthy is perhaps less their existence than the fact that we can calculate their interval structure and Whitney numbers.

We also establish the existence and properties of a second series of examples whose "universal" lattices $P_n^{\S}(\mathfrak{G})$, of which the others are join subsemilattices, consist of the $\mathfrak{G}$-valued and the ordinary partitions of an n-element set.

Each of these latter examples is the lattice of the lift matroid of a gain graph derived from $\mathfrak{G}$ and a graph, just as each of the first series is the lattice of the bias matroid. For the definitions of bias and lift matroids we refer the reader to Chapter I. For the definitions of $\mathfrak{G}$-valued (partial) partitions and their meet and join we refer to [1] or Chapter I; for the other definitions we refer to Chapter I. This chapter is devoted solely to the statement and proof of properties of examples.

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1. Dowling's lattice and its lift analog.

We begin with the "universal" examples of Theorems 3, 3, that is $P_n(\mathfrak{G})$, $P_n^\dagger(\mathfrak{G})$, and $P_n^\S(\mathfrak{G})$ for a group $\mathfrak{G}$. Most of the invariants of $P_n^\dagger(\mathfrak{G})$, Dowling's lattice, are in his article [1], but not the doubly indexed Whitney numbers.

Theorem 1A. [1]. *Let $n \geq 0$ be a cardinal number (finite or infinite) and $\mathfrak{G}$ a group.*

(I) The partially ordered set $P_n^\dagger(\mathfrak{G})$ of $\mathfrak{G}$-valued partial partitions of an n-set V is a finitary geometric lattice of rank n with meet and join as in Chapter I. Its bottom element o_n is o_V; its top element is $o_\varnothing$. Its rank function is $\operatorname{rk}\gamma = \sum\{|B|-1 : B \in \pi(\gamma)\}$, which equals $n - |\pi(\gamma)|$ if n is finite. It is the Dowling lattice $Q_n(\mathfrak{G})$.

The set $P_n(\mathfrak{G})$ of $\mathfrak{G}$-partitions of V is a lower order ideal. Its maximal elements all have corank one.

(II) If $\gamma \leq \delta$ in $P_n^\dagger(\mathfrak{G})$, let p be the number of blocks of $\pi(\gamma)$ lying in $(\operatorname{supp}\gamma)\backslash(\operatorname{supp}\delta)$, and for $D \in \pi(\delta)$ let p_D be the number of blocks of $\pi(\gamma)$ contained in D. Then the interval $[\gamma, \delta]$ is isomorphic to

$$P_p^\dagger(\mathfrak{G}) \times \bigtimes_{D\in\pi(\delta)} \Pi_{p_D}.$$

(III) Let n be a positive integer and let $\mathfrak{G}$ have finite order M. The Möbius invariant of $P_n^\dagger(\mathfrak{G})$ equals

$$(-1)^n((n-1)M+1)((n-2)M+1)\dots(M+1)(1),$$

its characteristic polynomial equals

$$M^n\left(\frac{y-1}{M}\right)_n,$$

its Whitney number polynomial is equal to

$$M^n\sum_{i=0}^n x^i\left(\frac{y-1}{M}\right)_{n-i}\sum_{l=n-i}^n\binom{n}{l}M^l S(l,n-i),$$

and its doubly indexed Whitney numbers are given by

$$w_{ij}(P_n^\dagger(\mathfrak{G})) = (-1)^{j-i}\sum_{l=n-i}^n M^l\binom{n}{l}S(l,n-i)\sum_{k=i}^j\binom{n-k}{n-j}|s(n-i,n-k)|.$$

Invariants of $P_n(\mathfrak{G})$ are its characteristic polynomial

$$p(P_n(\mathfrak{G});y) = M^{n-1}\left(\frac{y}{M}-1\right)_{n-1},$$

its Whitney number polynomial

$$w(P_n(\mathfrak{G});x,y) = M^{n-1}w_{\Pi_n}\left(x,\frac{y}{M}\right),$$

and the doubly indexed Whitney numbers

$$w_{ij}(P_n(\mathfrak{G})) = M^j w_{ij}(\Pi_n) = M^j S(n,n-i)s(n-i,n-j).$$

The Möbius function of an interval is

$$\mu(\gamma,\delta) = (-1)^{|\pi(\gamma)|-|\pi(\delta)|}M^p\left(p-1+\frac{1}{M}\right)_p\prod_{D\in\pi(\delta)}(p_D-1)!,$$

where p and p_D are defined in (II).

Proof. (I, II) See [1] for all but the remarks about $P_n(\mathfrak{G})$. Dowling's proof is general although he only treats finite n and $\mathfrak{G}$ explicitly.

These are also a special case of Theorem I.3.2A.

(III) Since $P_n(\mathfrak{G}) = \mathrm{Lat}^{\mathrm{b}}\,\mathfrak{G}K_n$ and $P_n^{\dagger}(\mathfrak{G}) = \mathrm{Lat}\,G(\mathfrak{G}K_n^{\bullet})$ by Lemma I.3.1A, we have the result from Section I.4. $\square$

Theorem 1B. *Let $n \geq 0$ be a cardinal number and $\mathfrak{G}$ a group.*

(I) The partially ordered set $P_n^{\S}(\mathfrak{G})$ is a finitary geometric lattice of rank n (but rank 1 if $n = 0$) with meet and join as above. Its bottom element is $o_n = o_V$ and its top element is the trivial partition 1_V. Its rank function is as in $P_n^{\dagger}(\mathfrak{G})$ for $\mathfrak{G}$-partitions and is $\mathrm{rk}\,\pi = 1 + \sum\{|B| - 1 : B \in \pi\} = n + 1 - |\pi|$ for partitions.

(II) For $\gamma \leq \pi$ in $P_n^{\S}(\mathfrak{G})$, where γ is a $\mathfrak{G}$-partition and π is a partition, let p_C be the number of blocks of $\pi(\gamma)$ in $C \in \pi$. Then

$$[\gamma, \pi] \cong \left(\bigtimes_{C \in \pi} P_{p_C}(\mathfrak{G}) \right) \cup \left(\bigtimes_{C \in \pi} \Pi_{p_C} \right).$$

(III) Let n be a positive integer and let $\mathfrak{G}$ have finite order M. The lattice $P_n^{\S}(\mathfrak{G})$ has Möbius invariant $(-1)^n M^{n-1}(n-1)!$, characteristic polynomial

$$p(P_n^{\S}(\mathfrak{G}); y) = (y-1)M^{n-1}\left(\frac{y}{M} - 1\right)_{n-1},$$

and Whitney number polynomial

$$x^n + \sum_{i=0}^{n-1} (y-1)x^i \{S(n, n-i)M^{n-1}\left(\frac{y}{M} - 1\right)_{n-i-1} + S(n, n-i+1)(y-2)_{n-i-1}\}.$$

The Möbius function of an interval $[\gamma, \pi]$ is

$$\mu(\gamma, \pi) = -(-M)^{|\pi(\gamma)| - |\pi|} \prod_{C \in \pi} (p_C - 1)!.$$

Proof. (I, II) are a simple case of Theorem I.3.1B.

(III) From Lemma I.3.1B and results in Section I.4, except that the last formula requires part (II) and the Frucht-Rota evaluation $\mu(\Pi_n) = (-1)^{n-1}(n-1)!$. $\square$

2. Set-restricted $\mathfrak{G}$-partitions

A variant of the Dowling lattice $P_n^{\dagger}(\mathfrak{G})$ is obtained by taking all partial $\mathfrak{G}$-partitions γ of the ground set V for which $|V \backslash \mathrm{supp}\,\gamma| \neq 1$. More generally let us divide V into two subsets, K of order k and L of order l, where $k + l = n$. Let $P_{(k,l)}^{\dagger}(\mathfrak{G})$ consist of all γ except those satisfying $V \backslash \mathrm{supp}\,\gamma = \{w\} \subseteq L$. Then

$$P_{(0,n)}^{\dagger}(\mathfrak{G}) = \{\gamma \in P_n^{\dagger}(\mathfrak{G}) : |V \backslash \mathrm{supp}\,\gamma| \neq 1\}.$$

The method of Section I.3 gives us the following result, which we leave rather incomplete since it is not in our main line of development.

Proposition 2A. *Let k, l be cardinal numbers and $\mathfrak{G}$ a group of order $M \geq 2$. Let K and L be disjoint sets of orders k and l and let $V = K \cup L$. Assume $n = k + l \geq 2$ or $k \geq 1$.*

(I) The set $P^{\dagger}_{(k,l)}(\mathfrak{G})$ is a geometric lattice of rank $n = k + l$. Its meet operation is given by

$$\gamma_1 \wedge \gamma_2, \text{ if } \gamma_1 \wedge \gamma_2 \in P^{\dagger}_{(k,l)}(\mathfrak{G});$$
$$(\gamma_1 \wedge \gamma_2) \cup o_{\{w\}}, \text{ if } \{w\} = V \backslash \operatorname{supp}(\gamma_1 \wedge \gamma_2) \subseteq L.$$

(II) Suppose $\gamma \leq \delta$ in $P^{\delta}_{(k,l)}(\mathfrak{G})$. The interval $[\gamma, \delta]$ is isomorphic to

$$P^{\dagger}_{(k^*,l^*)}(\mathfrak{G}) \times \bigtimes_{D \in \pi(\delta)} \Pi_{\pi(\gamma|_D)},$$

where

$$\begin{aligned} k^* &= \#\{B \in \pi(\gamma) : B \not\subseteq \operatorname{supp}\delta,\ |B| \geq 2\} \\ &\quad + \#\{\{w\} \in \pi(\gamma) : w \in K \backslash \operatorname{supp}\delta\}, \\ l^* &= \#\{\{w\} \in \pi(\gamma) : w \in L \backslash \operatorname{supp}\delta\}. \end{aligned}$$

(If in particular $k = 0$, then k^* is the number of nonsingleton blocks and l^* the number of singletons of $\gamma|_{\operatorname{supp}\gamma \backslash \operatorname{supp}\delta}$.)

(III) Suppose k, l, M are finite. The characteristic polynomial of $P^{\dagger}_{(k,l)}(\mathfrak{G})$ is

$$(y-1)(y-1-M)\cdots(y-1-(n-2)M)\cdot(y-(n-1)(M-1)-k),$$

the Möbius invariant is equal to

$$(-1)^n(M+1)(2M+1)\cdots((n-2)M+1)\cdot((n-1)(M-1)+k),$$

and the Whitney number polynomial is

$$w(P^{\dagger}_n(\mathfrak{G}); x, y) - l(x-1)(y-1)w(P_{n-1}(\mathfrak{G}); x, y-1).$$

Sketch of Proof. The basic fact is that $P^{\dagger}_{(k,l)}(\mathfrak{G}) = \operatorname{Lat} G(\mathfrak{G}K_n^{(K)})$, where $\Phi^{(K)}$ denotes Φ with a half edge added at each vertex in K. The meet can be deduced from this observation. The interval structure follows by contracting $\mathfrak{G}K_n^{(K)}$, noticing that contraction of an edge results in a vertex supporting an unbalanced loop, equivalent to a half edge. The invariants are worked out by means of the formulas in Section I.4. □

The Möbius function of an interval in $P^{\dagger}_{(0,n)}(\mathfrak{G})$, for instance, can be deduced from parts (II) and (III). We included all $P^{\dagger}_{(k,l)}$ in the proposition, not just the symmetrical case $P^{\dagger}_{(0,n)}$, for this reason.

The case $M = 1$ is omitted here because then $P^{\dagger}_{(k,l)}(\mathfrak{G})$ is the partition lattice of the graph composed of K_n with a new vertex adjacent to each member of K.

See Proposition 3 for a similar result where the partitions are further restricted.

3. Lattices based on a graph

In the balance of this chapter we study special types of $\mathfrak{G}$-valued partitions. A $\mathfrak{G}$-partition γ is *strict* if, whenever a constituent γ_B is trivial ($\pi(\gamma_B) = 1_B$), then B is a singleton. We let

$$P'^{\dagger}_n(\mathfrak{G}) = \text{ the set of strict partial } \mathfrak{G}\text{-partitions of an } n\text{-set},$$

with $P'_n(\mathfrak{G})$ = that of strict $\mathfrak{G}$-partitions and ${P'}^{\S}_n(\mathfrak{G}) = P'_n(\mathfrak{G}) \cup \Pi_n$. Let Γ be a simple graph on V of order n and γ a partial $\mathfrak{G}$-partition of V. We call γ Γ-*connected* if its underlying partition is Γ-connected, and we define $\mathbf{I}(\Gamma)$ by

$$\mathbf{I}_{vw}(\Gamma) = \begin{cases} \mathfrak{G} & \text{if } vw \in E(\Gamma), \\ \varnothing & \text{otherwise.} \end{cases}$$

We call γ Γ-*strict* if B is Γ-connected whenever γ_B is a trivial constituent, and we define $\mathbf{I}'(\Gamma)$ by

$$\mathbf{I}'_{vw}(\Gamma) = \begin{cases} \mathfrak{G} & \text{if } vw \in E(\Gamma), \\ \mathfrak{G}\backslash\{1\} & \text{if not.} \end{cases}$$

A $\mathfrak{G}$-partition γ is Γ-*linked* if for every constituent γ_B the edges of Γ between subblocks connect the subblocks; that is, if $\Gamma/\pi(\gamma_B)$ is connected. Let $\mathbf{I}^\circ(\Gamma)$ be defined by

$$\mathbf{I}^\circ_{vw}(\Gamma) = \begin{cases} \mathfrak{G} & \text{if } vw \in E(\Gamma), \\ \{1\} & \text{otherwise.} \end{cases}$$

Finally, γ is *strongly* Γ-*linked* if, for every constituent, $\Gamma\langle\pi(\gamma_B)\rangle$ is connected; that is, B is connected by the Γ-edges between blocks of B. We define $\mathbf{I}''(\Gamma)$ by

$$\mathbf{I}''_{vw}(\Gamma) = \begin{cases} \mathfrak{G}\backslash\{1\} & \text{if } vw \in E(\Gamma), \\ \varnothing & \text{otherwise.} \end{cases}$$

Each of these types of γ gives lattices of the form $P^\dagger(\mathfrak{G}, \mathbf{I})$ and $P^\S(\mathfrak{G}, \mathbf{I})$, hence whose rank function, bottom (and for $P^\dagger$ top) elements, and join are as in $P^\dagger_n(\mathfrak{G})$ and $P^\S_n(\mathfrak{G})$. So we need to describe in each case only the meet and, for $P^\S$, the top element, besides the always idiosyncratic interval structure and invariants. In most cases the meet of ordinary partitions in $P^\S$ will be as in Π_n; we mention it only when it differs from that. (Then it will be as in $\Pi(\Gamma)$, that is $\pi_1 \wedge \pi_2 \wedge \pi(\Gamma)$.)

To describe the interval structures of ${P'}^\dagger_n(\mathfrak{G})$ and ${P'}^\S_n(\mathfrak{G})$ we need sets of the form

$$P(\mathfrak{G}, \mathbf{I}'(K_p + \bar{K}_s)),\ P^\dagger(\text{same}), \text{ and } \Pi(K_p + \sum_{g \in \mathfrak{G}} \bar{K}_{s(g)}),$$

where $p, s, s(g) \geq 0$. The former are the sets of [partial] $\mathfrak{G}$-partitions of $[p] \cup [s]$ such that any trivial block contained in $[s]$ is a singleton. The latter is the set of partitions of $[p] \cup \bigcup_g [s(g)]$ such that any block contained entirely in some $[s(g)]$ is a singleton.

Let i' denote the natural injections ${P'}^\dagger_n \to P^\dagger_n$ and ${P'}^\S_n \to P^\S_n$. The adjoint maps (cf. (I.3.1)) are given by the following descriptions: ${i'}^\Delta(\gamma)$ is obtained from γ by breaking each trivial block B into the singletons $\{x\}$, $x \in B$; and ${i'}^\Delta(\pi) = \pi$.

Theorem 3A. *Let $n > 0$ be a cardinal number and $\mathfrak{G}$ a group of order at least two.*

(I) The set ${P'}^\dagger_n(\mathfrak{G})$ is a geometric lattice of rank n, isomorphic to $P^\dagger(\mathfrak{G}, \mathbf{I})$ with $\mathbf{I} \equiv \mathfrak{G}\backslash\{1\}$. Its meet is $\gamma_1 \wedge' \gamma_2 = {i'}^\Delta(\gamma_1 \wedge \gamma_2)$. The set $P'_n(\mathfrak{G}) \cong P(\mathfrak{G}, \mathbf{I})$ is a lower ideal whose maximal elements have corank 1.

(II) Suppose $\gamma \leq \delta$ in ${P'}^\dagger_n(\mathfrak{G})$. For $Y \subseteq \operatorname{supp}\gamma$ let p_Y = the number of nonsingleton blocks and s_Y = the number of singleton blocks of γ in Y. For $D \in \pi(\delta)$ and $g \in \mathfrak{G}$ let $s_D(g)$ =

the number of singleton blocks $\{x\}$ *of* $\gamma|_D$ *for which* $\delta_D(x) = g$*. Let* $X = (\operatorname{supp}\gamma)\backslash(\operatorname{supp}\delta)$*. The interval* $[\gamma, \delta]$ *is isomorphic to*

$$P^\dagger(\mathfrak{G}, \mathbf{I}'(K_{p_X} + \bar{K}_{s_X})) \times \bigtimes_{D\in\pi(\delta)} \Pi(K_{p_D} + \sum_{g\in\mathfrak{G}} \bar{K}_{s_D(g)}).$$

(III) Let n *and the order* M *of* $\mathfrak{G}$ *be finite. The polynomials of* $P'_n(\mathfrak{G})$ *are*

$$p(P'_n(\mathfrak{G}); y) = \sum_{k=1}^{n} S(n,k) M^{k-1} \left(\frac{y}{M} - 1\right)_{k-1}$$

and

$$w(P'_n(\mathfrak{G}); x, y) = \sum\sum\sum_{I+J+K=n,\ I,J,K\geq 0} \binom{n}{I, J, K} \sum_{i=0}^{I} S(I,i)(-M)^i x^{I-i}$$
$$\cdot \sum_{j=0}^{J} S(J,j) M^J x^{J-j} \sum_{\substack{k=0\\ i+j+k\geq 1}}^{K} S(K,k) M^{k-1} \left(\frac{y}{M} - 1\right)_{i+j+k-1},$$

and the Whitney numbers are

$$w_r = \sum_{k=0}^{r} S(n, n-k) s(n-k, n-r) M^{r-k},$$
$$W_r = \sum\sum_{\substack{I+J\leq n\\ I,J\geq 0}} \binom{n}{J}\binom{n-J}{I} \sum_{i=0}^{I}\sum_{\substack{j=0\\ I-i+J-j=n-r}}^{J} (-1)^i S(I,i) S(J,j) M^{J-j},$$

all provided $n \geq 1$*.*

The polynomials of ${P'}^\dagger_n(\mathfrak{G})$ *are*

$$p({P'}^\dagger_n(\mathfrak{G}); y) = \sum_{k=0}^{n} S(n,k) M^k \left(\frac{y-1}{M}\right)_k,$$
$$w({P'}^\dagger_n(\mathfrak{G}); x, y) = \sum_{m=0}^{n} \binom{n}{m} x^{n-m} w(P'_m(\mathfrak{G}); x, y-1).$$

Its Möbius invariant is

$$\mu({P'}^\dagger_n(\mathfrak{G})) = \sum_{k=0}^{n} (-1)^k S(n,k)(1)(1+M)(1+2M)\cdots(1+(k-1)M).$$

The Whitney numbers of ${P'}^\dagger_n(\mathfrak{G})$ *are*

$$w_r = (-1)^r \sum_{k=n-r}^{n} S(n,k) \sum_{i=n-r}^{k} M^{k-i} s(k,i) \binom{i}{n-r} (-1)^{n-i},$$
$$W_r = \sum\sum_{i,j\geq 0,\ i+j\leq n-r} (-1)^i \binom{n}{r+i+j} \sum_{J=j}^{r+j} \binom{r+i+j}{J} S(J,j) S(r+i+j+1-J, i+1) M^{J-j}.$$

The Möbius function of an interval $[\gamma, \delta]$ *in* $P'^{\dagger}_n(\mathfrak{G})$ *is*

$$\sum_{k=s_X}^{p_X+s_X} (-1)^k S(p_X, k-s_X) M^k \left(k-1+\frac{1}{M}\right)_k \prod_{D\in\pi(\delta)} \mu(\Pi(K_{p_D} + \sum_{g\in\mathfrak{G}} \bar{K}_{s_D(g)})).$$

Since the graph in the last factor is complete multipartite, its Möbius invariant is easy to compute. Recall that it is

$$\sum_{k=0}^{s_D} (-1)^{p_D+k-1}(p_D+k-1)! \sum_{\sum i(g)=k,\ 0\le i(g)\le s_D(g)} \prod_{g\in\mathfrak{G}} S(s_D(g), i(g)).$$

Proof. Let $\mathbf{I} \equiv \mathfrak{G}\backslash\{1\}$. Clearly, for any partial $\mathfrak{G}$-partition γ, $\Delta(\gamma_B)$ is connected if γ_B is nontrivial and is $\bar{K}_B$ in the contrary case. So γ is in $P^{\dagger}(\mathfrak{G}, \mathbf{I})$ precisely when it is strict. We also see from this observation that the meet operation is as described.

We could simply have referred to Theorem 3 below with $\Gamma = \bar{K}_n$, as we do for the rest of the proof.

(II) This is immediate from Theorem 3(II) since $s_X = |\sigma_X|$, etc.

(III) The characteristic polynomials are from Theorem 3(III) since $\psi_k(\bar{K}_n) = S(n, k)$. But the Whitney number polynomials have to be worked out here because we do not give them in the more general situation of Γ-strict $\mathfrak{G}$-partitions.

The gain graph underlying $P'_n(\mathfrak{G})$ is $\Phi = (\mathfrak{G}\backslash\{1\})K_n$. We determine its Whitney number polynomial by counting zero-free l-colorings $\kappa\lambda$ of Φ. We need the rank of $I(\kappa\lambda)$. It suffices to examine a set $B = \kappa^{-1}(m)$, $1 \le m \le l$. The improper edges are those linking $v, w \in B$ on which λ has different values. Thus the improper edge set (on B) has rank $|B| - 1$ if λ is nonconstant and otherwise rank 0. We conclude that

$$\begin{aligned}
w^*_{\varphi}(x, lM) &= \sum_{\pi\in\Pi_n} (l)_{|\pi|} \prod_{B\in\pi} \{(M^{|B|} - m)x^{|B|-1} - Mx^0\} \\
&= \sum_{X\cup Y\cup Z=V} \sum_{\rho\in\Pi_X,} \sum_{\sigma\in\Pi_Y,} \sum_{\tau\in\Pi_Z} X^{|X\cup Y|-|\rho\cup\sigma|} (-1)^{|\rho|} M^{|\rho|+|Y|+|\tau|} (l)_{|\rho\cup\sigma\cup\tau|} \\
&= \sum_{\substack{I+J+K=0\\ I,J,K\ge 0}} \binom{n}{I, J, K} \sum_{i=0}^{I} S(I, i)(-M)^i \sum_{j=0}^{J} S(J, j)M^J \\
&\quad \cdot \sum_{k=0}^{K} S(K, k)M^k x^{I-i+J-j} (l)_{i+j+k}.
\end{aligned}$$

This gives $w(P'_n(\mathfrak{G}); x, y)$ by setting $l = y/M$ and dividing by y, since the semilattice has rank $n - 1$.

The rest of the calculations are routine, except for the Möbius function. This comes from part (II) and the evaluation of $\mu(P^{\dagger}(\mathfrak{G}, \mathbf{I}'(\Gamma)))$ in Theorem 3(III) for $\Gamma = K_p + \bar{K}_s$. Then $\psi_k(\Gamma) = S(s, k - p)$, which is zero for $k < p$. □

Theorem 3B. *Let* n *and* $\mathfrak{G}$ *be as in Theorem 3.*

(I) The set $P'^{\S}_n(\mathfrak{G})$ *is a geometric lattice of rank* n*, isomorphic to* $P^{\S}(\mathfrak{G}, \mathbf{I})$ *with* $\mathbf{I} \equiv \mathfrak{G}\backslash\{1\}$*. The meet of a partition* π *and a strict* $\mathfrak{G}$*-partition* γ *is* $\gamma \wedge' \pi = i'^{\Delta}(\gamma \wedge \pi)$*. The top element is the trivial partition.*

(II) An interval $[\gamma,\pi]$ *of* $P'^{\S}_n(\mathfrak{G})$ *is isomorphic to*

$$\left(\bigtimes_{C\in\pi} P(\mathfrak{G},\mathbf{I}'(K_{p_C}+\bar{K}_{s_C}))\right)\cup\left(\bigtimes_{C\in\pi}\Pi_{p_C+s_C}\right),$$

with definitions given in Theorem 3(II).

(III) Let n *be finite and let* $M=|\mathfrak{G}|$ *also finite. The Möbius invariant is*

$$\mu(P'^{\S}_n(\mathfrak{G}))=\sum_{k=1}^{n}(-1)^k(k-1)!M^{k-1}S(n,k).$$

The polynomials are

$$p(P'^{\S}(\mathfrak{G});y)=(y-1)\sum_{k=1}^{n}S(n,k)M^{k-1}\left(\frac{y}{M}-1\right)_{k-1},$$

$$w(P'^{\S}_n(\mathfrak{G});x,y)=(y-1)w(P_n(\mathfrak{G});x,y)+xw_{\Pi_n}(x,y).$$

The Whitney numbers are

$$w_r=w_r(P'_n(\mathfrak{G}))+(-1)^r\sum_{k=n-r}^{n-1}\sum_{i=n-r}^{k}M^{k-(n-r)}S(n,k+1)s(k,i)\binom{i}{n-r}(-1)^{n-1-i},$$

$$W_r=W_r(P'_n(\mathfrak{G}))+S(n,n+1-r).$$

The Möbius function of an interval $[\gamma,\pi]$ *equals*

$$-\prod_{C\in\pi}\sum_{k=s_C}^{p_C+s_C}(-M)^{k-1}(k-1)!S(p_C,k-s_C).$$

Proof. We rely on Theorem 3 below with $\Gamma=\bar{K}_n$ for parts (I, II) and on Theorem 3(III). The Möbius function of an interval is calculated by combining part (II) with the reduction of $\mu(P^{\S})$ to an evaluation of $p(P;y)$. Here P is a product of which each factor has the form $P(\mathfrak{G},\mathbf{I}'(\Gamma))$; so we can find the characteristic polynomial by multiplying results from Theorem 3(III). □

The strict partial $\mathfrak{G}$-partitions for which $|V\backslash\operatorname{supp}\gamma|\neq 1$ also form a geometric lattice. Analogous to Proposition 2 we have

Proposition 4A. *Let* $k,l,\mathfrak{G}$*, etc., be as in Proposition 2, but with* $n=k+l\geq 3$*, or else* $n=2$ *and either* $M\geq 3$ *or* $k>0$*.*

(I) The set $P'^{\dagger}_{(k,l)}(\mathfrak{G})=P'^{\dagger}(\mathfrak{G})\cap P^{\dagger}_{(k,l)}(\mathfrak{G})$ *is a geometric lattice of rank* n*. Its meet is given by*

$$\gamma_1\wedge'\gamma_2,\quad \text{if}\ \ \gamma_1\wedge'\gamma_2\in P'^{\dagger}_{(k,l)}(\mathfrak{G}),$$

$$(\gamma_1\wedge'\gamma_2)\cup o_{\{w\}},\quad \text{if}\ \ \{w\}=V\backslash\operatorname{supp}(\gamma_1\wedge'\gamma_2)\subseteq L.$$

(III) Invariants are

$$\mu(P'^{\dagger}_{(k,l)}(\mathfrak{G}))=\sum_{i=0}^{n}(-1)^i(1)(M+1)(2M+1)\cdots((i-1)M+1)\{S(n,i)+lS(n-1,i)\},$$

$$p(P'^{\dagger}_{(k,l)}(\mathfrak{G});y)=\sum_{i=0}^{n}M^i\left(\frac{y-1}{M}\right)_i\{S(n,i)+lS(n-1,i)\}.$$

Sketch of proof. The lattice is isomorphic to $\operatorname{Lat} G([\mathfrak{G}\backslash\{1\}]K_n^{(K)})$. The meet and invariants follow easily. ☐

The interval structure can be calculated; it is like that in ${P'}_n^{\dagger}(\mathfrak{G})$ except that the first factor is more complicated and has a different form depending on whether $M = 2$ or $M \geq 3$ (since in the former case an induced subgraph of order two can be balanced).

Of all the lattices based on a graph the simplest are those of $\mathfrak{G}$-valued Γ-partitions. They are almost as well behaved as $P_n^{\dagger}(\mathfrak{G})$ and $P_n^{\S}(\mathfrak{G})$ themselves and are valuable for later examples.

The adjoints of the natural injections $i_{\mathbf{I}(\Gamma)}$ are $i^{\Delta}_{\mathbf{I}(\Gamma)}(\gamma)$ and $i^{\Delta}_{\mathbf{I}(\Gamma)}(\pi)$ obtained from γ, or π, by replacing each block B by its $\Gamma{:}B$-connected components.

Theorem 5A. *Let Γ be a simple graph on vertex set V of order n, a cardinal number, and let $\mathfrak{G}$ be any group.*

(I) The set of Γ-connected partial $\mathfrak{G}$-partitions of V is the geometric lattice $P^{\dagger}(\mathfrak{G}, \mathbf{I}(\Gamma))$, of rank n. Its meet is

$$\gamma_1 \wedge_\Gamma \gamma_2 = i^{\Delta}_{\mathbf{I}(\Gamma)}(\gamma_1 \wedge \gamma_2).$$

The set of Γ-connected $\mathfrak{G}$-partitions of V is the lower ideal $P(\mathfrak{G}, \mathbf{I}(\Gamma))$, whose maximal elements have corank $c(\Gamma)$.

(II) An interval $[\gamma, \delta]$ in $P^{\dagger}(\mathfrak{G}, \mathbf{I}(\Gamma))$ is isomorphic to

$$P^{\dagger}(\mathfrak{G}, \mathbf{I}(\Gamma/\pi(\gamma|_X))) \times \bigtimes_{D \in \pi(\delta)} \Pi(\Gamma/\pi(\gamma|_D)),$$

where $X = (\operatorname{supp} \gamma)\backslash(\operatorname{supp} \delta)$.

(III) Let n be finite and let $\mathfrak{G}$ have finite order M.

The characteristic polynomial of $P(\mathfrak{G}, \mathbf{I}(\Gamma))$ is $M^n \chi_\Gamma(y/M)/y^{c(\Gamma)}$. The Whitney number polynomial is $M^n w_\Gamma(x, y/M)/y^{c(\Gamma)}$.

The characteristic polynomial of $P^{\dagger}(\mathfrak{G}, \mathbf{I}(\Gamma))$ equals $M^n \chi_\Gamma((y-1)/M)$ and its Whitney number polynomial is

$$\sum_{X \subseteq V} x^{n-|X|} M^{|X|} w_{\Gamma X}\left(x, \frac{y-1}{M}\right).$$

The Möbius invariant is equal to $M^n \chi_\Gamma(-1/M)$. The doubly indexed Whitney numbers are

$$w_{ij} = \sum_{\substack{X \subseteq V \\ m=|X| \geq n-i}} M^m w_{i+m-n, j+m-n}(\Gamma{:}X).$$

The Möbius function of an interval $[\gamma, \delta]$ equals

$$M^{|\pi(\gamma|_X)|} \chi_{\Gamma/\pi(\gamma|_X)}(-M^{-1}) \cdot \prod_{D \in \pi(\delta)} \mu(\Pi(\Gamma/\pi(\gamma|_D))).$$

Proof. (I) Consider $\mathbf{I} = \mathbf{I}(\Gamma)$. For any γ, clearly $\Delta(\gamma_B) \cong \Gamma{:}B$. Thus $P^{\dagger}(\mathfrak{G}, \mathbf{I}(\Gamma))$ and $P(\mathfrak{G}, \mathbf{I}(\Gamma))$ are the sets of Γ-connected (partial) $\mathfrak{G}$-partitions. Since $\Delta_{\mathbf{I}(\Gamma)} = \Gamma$, the rest follows from Theorem I.3.1A.

(II) This follows from Theorem I.3.1A without any difficulty.

(III) This follows from results of Section I.4, since the underlying gain graph is $\mathfrak{G}\Gamma^{\bullet}$. ☐

Theorem 5B. *Let Γ and $\mathfrak{G}$ be as in Theorem 3.*

(I) The set of Γ-connected $\mathfrak{G}$-partitions and ordinary Γ-connected partitions of V is the rank $n+1-c(\Gamma)$ geometric lattice $P^{\S}(\mathfrak{G}, \mathbf{I}(\Gamma))$. Its top element is $\pi(\Gamma)$. The meet of a Γ-connected $\mathfrak{G}$-valued partition γ and an ordinary partition π is $\gamma \wedge_\Gamma \pi = i^{\Delta}_{\mathbf{I}(\Gamma)}(\gamma \wedge \pi)$. The meet of ordinary partitions is as in $\Pi(\Gamma)$.

(II) An interval $[\gamma, \pi]$ is isomorphic to $P^{\S}(\mathfrak{G}, \mathbf{I}([\Gamma{:}\pi]/\pi(\gamma)))$.

(III) Let n and $M = |\mathfrak{G}|$ be finite. The characteristic polynomial of $P^{\S}(\mathfrak{G}, \mathbf{I}(\Gamma))$ is equal to $(y-1)M^n\chi_\Gamma(y/M)/y^{c(\Gamma)}$. Its Whitney number polynomial is

$$\{M^n(y-1)w_\Gamma\left(x, \frac{y}{M}\right) + xw_\Gamma(x,y)\}/y^{c(\Gamma)}.$$

The Möbius invariant equals $-M^{n-c(\Gamma)}\mu(\Pi(\Gamma))$ and in general the doubly indexed Whitney numbers are

$$w_{ij} = M^j w_{ij}(\Gamma) - M^{j-1}w_{i,j-1}(\Gamma) + w_{i-1,j-1}(\Gamma).$$

Proof. Immediate from Sections I.3 and I.4 as indicated by the proof of Theorem 3. □

The natural injections $i_{\mathbf{I}'(\Gamma)}$ have adjoints given by

$$i^{\Delta}_{\mathbf{I}'(\Gamma)}(\gamma) = \gamma \wedge \{\pi(\Gamma{:}\pi(\gamma)) \vee \pi(i'^{\Delta}(\gamma))\},$$
$$i^{\Delta}_{\mathbf{I}'(\Gamma)}(\pi) = \pi.$$

The former is obtained from γ by dividing each trivial block B into its $\Gamma{:}B$-connected components, with constituents equal to the restrictions of γ_B.

Recall that the coefficients $\psi_k(\Gamma)$ are defined by $\chi_\Gamma(y) = \sum_0^n \psi_k(\Gamma)(y)_k$.

Theorem 6A. *Let Γ be a simple graph on vertex set V of order $n > 0$, a cardinal number, and let $\mathfrak{G}$ be a group of order $M \geq 2$.*

(I) The set of Γ-strict partial $\mathfrak{G}$-partitions of V is the rank n geometric lattice $P^{\dagger}(\mathfrak{G}, \mathbf{I}'(\Gamma))$, whose meet is $\gamma_1 \wedge'_\Gamma \gamma_2 = i^{\Delta}_{\mathbf{I}'(\Gamma)}(\gamma_1 \wedge \gamma_2)$.

The set of Γ-strict $\mathfrak{G}$-partitions corresponds to $P(\mathfrak{G}, \mathbf{I}'(\Gamma))$, which is a lower ideal whose maximal elements have corank one.

(II) Suppose $\gamma \leq \delta$ in $P^{\dagger}(\mathfrak{G}, \mathbf{I}'(\Gamma))$. For $Y \subseteq \operatorname{supp}\gamma$ let $\sigma_Y = \{B \in \pi(\gamma|_Y) : B$ is trivial$\}$ and $q_Y = |\pi(\gamma|_Y)\backslash\sigma_Y|$. For $D \in \pi(\delta)$ and $g \in \mathfrak{G}$ let $\sigma_D(g) = \{B \in \pi(\gamma|_D) : \gamma_B$ is trivial and $\delta_D(B) = g\}$. Let $X = (\operatorname{supp}\gamma)\backslash(\operatorname{supp}\delta)$. The interval $[\gamma, \delta]$ is isomorphic to

$$P^{\dagger}(\mathfrak{G}, \mathbf{I}(\Gamma/\sigma_X + K_{q_X})) \times \bigtimes_{D\in\pi(\delta)} \Pi(K_{q_D} + \sum_{g\in\mathfrak{G}} \Gamma/\sigma_D(g)).$$

(III) Assume n and M are finite. The characteristic polynomial of $P(\mathfrak{G}, \mathbf{I}'(\Gamma))$ is

$$\sum_{k=1}^{n} \psi_k(\Gamma)M^{k-1}\left(\frac{y}{M} - 1\right)_{k-1}.$$

The characteristic polynomial of $P^{\dagger}(\mathfrak{G}, \mathbf{I}'(\Gamma))$ equals

$$\sum_{k=1}^{n} \psi_k(\Gamma)M^k\left(\frac{y-1}{M}\right)_k$$

and its Möbius invariant is

$$\sum_{k=1}^{n}(-1)^k \psi_k(\Gamma)(1)(1+M)(1+2M)\cdots(1+(k-1)M).$$

Proof. (I) For $\mathbf{I} = \mathbf{I}'(\Gamma)$, clearly $\Delta_{\mathbf{I}} = K_n$; and clearly $\Delta(\gamma_B) = [\Gamma{:}\pi(\gamma_B)] \cup K_n\langle\pi(\gamma_B)\rangle$ for any partial $\mathfrak{G}$-partition γ. Thus $\Delta(\gamma_B)$ is connected if γ_B is nontrivial or if it is trivial and $\Gamma{:}B$ is connected, but not otherwise. That proves $\mathbf{I}'(\Gamma)$ gives the right set of $\mathfrak{G}$-partitions. The rest is immediate from Theorem I.3.1A and these observations, save for the formula for the meet. As to that, the partition in the braces is precisely the correct underlying partition for $\gamma_1 \wedge'_\Gamma \gamma_2$, since $B \in \pi(\gamma_1 \wedge \gamma_2)$ is partitioned into Γ-connected components by the left partition and is unpartitioned if nontrivial, totally partitioned if trivial, by the right partition.

(II) In view of Theorem I.3.1A(II) we have to calculate $\mathbf{I}^*$ of that theorem. Choose basepoints $v_B \in B$ and representatives γ_B so that $\gamma_B(v_B) = 1$. Consider A and $B \in \pi(\gamma)$. Clearly $\mathbf{I}^*_{AB} \supseteq \mathfrak{G}\backslash\{1\}$, and if γ_A or γ_B assumes a value other than 1 then $\mathbf{I}^*_{AB}$ contains (if $\gamma_A(v) = g \neq 1$, say) $g\mathbf{I}_{vv_B}$, which contains 1. Therefore, $\mathbf{I}^*_{AB} = \mathfrak{G}$ if either γ_A or γ_B is nontrivial. If both are trivial then $\mathbf{I}^*_{AB} = \mathfrak{G}$ precisely when there is a Γ-edge between A and B. Thus $\mathbf{I}^* = \mathbf{I}'(\Gamma/\sigma_V + K_{q_V})$, from which we get the first factor in part (II) by restricting to X instead of V.

For the second factor, in each D, we have to ask when A, $B \in \pi(\gamma|_D)$ are joined by a link in $\Delta(\delta_D)$. They certainly are if there exist $v \in A$ and $w \in B$ with $\delta_D(v) \neq \delta_D(w)$. Thus only if $\delta_D\big|_A$ and $\delta_D\big|_B$ are constant, that is to say if γ_A and γ_B are trivial, and moreover if $\delta_D(A) = \delta_D(B) = g$, might A and B not be djoined. In this latter case everything depends on Γ. Hence on each set $\sigma_D(g)$ the graph for the second factor is $\Gamma/\sigma_D(g)$, while every other pair $A, B \in \pi(\gamma|_D)$ are adjacent.

(III) We look at $\Psi = (\mathfrak{G}\backslash\{1\})K_n \cup \{1\}\Gamma$, since by Lemma I.3.1A it is $G(\Psi^\bullet)$ that has lattice $P^\dagger(\mathfrak{G}, \mathbf{I}'(\Gamma))$. Applying formula (I.4.2) to Ψ with $\Gamma = \Gamma_2 = K_n$ and with $\Gamma_1 =$ the present Γ gives

$$\chi^*_\Psi(y) = \sum_{\substack{\pi\in\Pi_n \\ E(\Gamma\!:\!\pi)=\varnothing}} M^{|\pi|}\left(\frac{y}{M}\right)_{|\pi|} = \sum_{k=0}^{n} \psi_k(\Gamma) M^k \left(\frac{y}{M}\right)_k.$$

Part (III) now follows routinely by results in Section I.4. $\square$

One can evaluate $w^*_\Psi(x, y)$ and from it the (semi)lattice Whitney number polynomials by means of Proposition I.4.4. To do so one needs $w^*_{\Psi{:}B}(x, M)$. This is easily determined. Consider $\lambda : B \to \mathfrak{G}$. If λ is nonconstant, then $I(\lambda)$ connects all of B; thus $\operatorname{rk} I(\lambda) = |B| - 1$. If λ is constant, $I(\lambda) = E(\Gamma{:}B)$. Therefore,

$$w^*_{\Psi{:}B}(x, M) = (M^{|B|} - M)x^{|B|-1} + Mx^{\operatorname{rk} G(\Gamma{:}B)}. \tag{1}$$

We drop the matter here because we have no particular need for the results elsewhere.

Theorem 6B. *Let Γ, $\mathfrak{G}$, etc., be as in Theorem 3.*

(I) The set of Γ-strict $\mathfrak{G}$-partitions and all ordinary partitions of V is the geometric lattice $P^\S(\mathfrak{G}, \mathbf{I}'(\Gamma))$, of rank n. Its top element is 1_V and its meet of mixed type is $\gamma \wedge'_\Gamma \pi = i^\Delta_{\mathbf{I}'(\Gamma)}(\gamma \wedge \pi)$.

(II) An interval $[\gamma,\pi]$ in $P^{\S}(\mathfrak{G},\mathbf{I}'(\Gamma))$ is isomorphic to

$$\left(\bigtimes_{C\in\pi} P(\mathfrak{G},\mathbf{I}'(\Gamma/\sigma_C+K_{q_C}))\right)\cup\left(\bigtimes_{C\in\pi}\Pi_{\pi(\gamma|_C)}\right),$$

σ_C and q_C being as in Theorem 3.

(III) Let n and M be finite. The characteristic polynomial of $P^{\S}(\mathfrak{G},\mathbf{I}'(\Gamma))$ is

$$(y-1)\sum_{k=1}^{n}\psi_k(\Gamma)M^{k-1}\left(\frac{y}{M}-1\right)_{k-1}$$

and its Möbius invariant equals $\sum_1^n(-1)^kM^{k-1}(k-1)!\psi_k(\Gamma)$.

Proof. (I) As in the proof of Theorem 3(I).

(II) From Theorem I.3.1B(II) and remarks in the proof of Theorem 3.

(III) From Theorem 3(III) and results in Section I.4. □

For the next results we need the adjoints of the natural injections $i_{\mathbf{I}^\circ(\Gamma)}$. They are

$$i^{\Delta}_{\mathbf{I}^\circ(\Gamma)}(\gamma)=\gamma\wedge\{[\pi(\Gamma)\wedge\pi(\gamma)]\wedge\pi_0(\gamma)\},$$
$$i^{\Delta}_{\mathbf{I}^\circ(\Gamma)}(\pi)=\pi.$$

The former is obtained from γ through replacing each constituent γ_B by its restrictions to the components of $[\Gamma:B]\cup[K_n:\pi(\gamma_B)]$, in other words to the supports of the blocks of $\pi(\Gamma/\pi(\gamma)_B)$.

Theorem 7A. *Let Γ be a simple graph on vertex set V of order $n>0$, not necessarily finite, and $\mathfrak{G}$ a group of order M.*

(I) The set of Γ-linked partial $\mathfrak{G}$-partitions of V is the rank n geometric lattice $P^{\dagger}(\mathfrak{G},\mathbf{I}^\circ(\Gamma))$. The meet $\gamma_1\wedge^\circ_\Gamma\gamma_2$ equals $i^{\Delta}_{\mathbf{I}^\circ(\Gamma)}(\gamma_1\wedge\gamma_2)$.

The set of Γ-linked $\mathfrak{G}$-partitions of V is the order ideal $P(\mathfrak{G},\mathbf{I}^\circ(\Gamma))$; its maximal elements have corank one.

(II) Let $\gamma\le\delta$ in $P^{\dagger}(\mathfrak{G},\mathbf{I}^\circ(\Gamma))$. Let $\mathbf{I}^*:\pi(\gamma)\times\pi(\gamma)\to\mathcal{P}(\mathfrak{G})$ be defined by $\mathbf{I}^*_{AB}=\mathfrak{G}$ if A and B are adjacent in $\Gamma/\pi(\gamma)$, otherwise $\mathbf{I}^*_{AB}=\Im\gamma_A)(\Im\gamma_B)^{-1}$ (where fixed representatives are chosen for all γ_B). For $D\in\pi(\delta)$ let Γ_D be the graph on $\pi(\gamma|_D)$ in which AB is an edge if $\delta_D(A)\cap\delta_D(B)\neq\varnothing$ or AB is an edge of $\Gamma/\pi(\gamma|_D)$. The interval $[\gamma,\delta]$ is isomorphic to

$$P^{\dagger}(\mathfrak{G},\mathbf{I}^*|\pi(\gamma|_X))\times\bigtimes_{D\in\pi(\delta)}\Pi(\Gamma_D).$$

(III) Let n and M be finite.

The Whitney number polynomial of $P(\mathfrak{G},\mathbf{I}^\circ(\Gamma))$ equals

$$\sum_{\pi\in\Pi_n}(M)_{|\pi|}y^{-1}w_{(K_n\bar{\pi})\cup\Gamma}\left(x,\frac{y}{M}\right).$$

The characteristic polynomial is the same with $w(x,y/M)$ changed to $\chi(y/M)$.

The Whitney number polynomial of $P^{\dagger}(\mathfrak{G},\mathbf{I}^\circ(\Gamma))$ equals

$$\sum_{X\subseteq V}\sum_{\pi\in\Pi_X}(M)_{|\pi|}\,x^{n-|X|}w_{(K_n\bar{\pi})\cup(\Gamma X)}\left(x,\frac{y-1}{M}\right).$$

The characteristic polynomial is

$$\sum_{\pi\in\Pi_n} (M)_{|\pi|}\ \chi_{(K_n/\pi)\cup\Gamma}\left(\frac{y-1}{M}\right)$$

and the Möbius invariant is obtained by setting $y = 0$.

Proof. (I) Let $\mathbf{I} = \mathbf{I}^\circ(\Gamma)$ and consider any partial $\mathfrak{G}$-partition γ. Evidently $\Delta(\gamma_B) = [K_n{:}\pi(\gamma_B)] \cup [\Gamma{:}B]$; it is connected precisely when γ_B is Γ-linked. Thus we have the right $\mathbf{I}$ and meet. Notice that $\Delta_{\mathbf{I}} = K_n$. Let Φ be the gain graph underlying $P(\mathfrak{G}, \mathbf{I}^\circ(\Gamma))$.

(II) In $[\gamma, 1]$, what should be $\mathbf{I}^*$ to give isomorphism with $P^\dagger(\mathfrak{G}, \mathbf{I}^*)$? Suppose an edge of Γ links A and $B \in \pi(\gamma)$. Then clearly $\mathbf{I}^*_{AB} = \mathfrak{G}$. If no such edge links them, then the only edges vw with $v \in A$, $w \in B$ have gains 1. Thus after switching Φ we get gains $\gamma_A(v_A)^{-1}\gamma_A(v)\gamma_B(w)^{-1}\gamma_B(v_B)$, where v_B is the arbitrary basepoint in B. The $\mathbf{I}^*_{AB}$ of Theorem I.3.1A is the set of all such values. If we switch by the inverted arbitrary basepoint evaluations, $f(B) = \gamma_B(v_B)^{-1}$, we get the $\mathbf{I}^*$ of the present theorem.

(III) What we need is the zero-free Whitney number polynomial of Φ. Let $\kappa\lambda$ be a zero-free k-coloring of Φ. Its set of improper edges (ignoring gain values) is

$$\begin{aligned} I_\Phi(\kappa\lambda) &= \{vw : vw \in I_\Gamma(\kappa) \ \text{ or } \ \kappa\lambda(v) = \kappa\lambda(w)\} \\ &= I_{\Delta(\pi)}(\kappa), \end{aligned}$$

where $\pi = \pi(\lambda)$ is the partition $\{\lambda^{-1}(g) \neq \varnothing : g \in \Im\lambda\}$ of V and thus $\Delta(\pi) = [K_n{:}\pi] \cup \Gamma$. If we choose π first there are $(M)_{|\pi|}$ possible choices for λ. Thus

$$\begin{aligned} w^*_\Phi(x, kM) &= \sum_{\kappa\lambda} x^{\operatorname{rk} I_\Phi(\kappa\lambda)} = \sum_{\pi\in\Pi_n} (M)_{|\pi|} \sum_{\kappa} x^{\operatorname{rk} I_{\Delta(\pi)}(\kappa)} \\ &= \sum_{\pi\in\Pi_n} (M)_{|\pi|} w_{\Delta(\pi)}(x, k). \end{aligned}$$

Part (III) now follows by the results of Section I.4. □

Theorem 7B. *Let Γ, $\mathfrak{G}$, etc., be as in part A.*

(I) The set of Γ-linked $\mathfrak{G}$-partitions and all ordinary partitions of V is the geometric lattice $P^\S(\mathfrak{G}, \mathbf{I}^\circ(\Gamma))$, of rank n. Its top element is 1_V. The meet of a mixed pair of elements is $\gamma \wedge^\circ_\Gamma \pi = i^\Delta_{\mathbf{I}^\circ(\Gamma)}(\gamma \wedge \pi)$.

(II) Let $\gamma \leq \pi$ in $P^\S(\mathfrak{G}, \mathbf{I}^\circ(\Gamma))$. The interval $[\gamma, \pi]$ is isomorphic to

$$\left(\bigtimes_{C\in\pi} P(\mathfrak{G}, \mathbf{I}^*|\pi(\gamma|_C))\right) \cup \left(\bigtimes_{C\in\pi} \Pi_{\pi(\gamma|_C)}\right).$$

(III) Let n and M be finite. The Whitney number polynomial of $P^\S(\mathfrak{G}, \mathbf{I}^\circ(\Gamma))$ is

$$\sum_{\pi\in\Pi_n} (M)_{|\pi|}\ \frac{y-1}{y}\ w_{(K_n/\pi)\cup\Gamma}\left(x, \frac{y}{M}\right),$$

its characteristic polynomial is obtained by putting $\chi(y/M)$ for $w(x, y/M)$, and its Möbius invariant equals

$$-\sum_{\pi} (M)_{|\pi|} \mu(\Pi([K_n{:}\pi] \cup \Gamma)),$$

summed over $\pi \in \Pi_n$ for which Γ/π is connected.

Proof. Standard. For the Möbius invariant notice that the contribution of π when $x = y = 0$ is zero unless $(K_n{:}\pi) \cup \Gamma$ is connected, when it is the indicated value μ. $\square$

The set $Q = P'^{\dagger}_n(\mathfrak{G}) \cap P^\dagger(\mathfrak{G}, \mathbf{I}(\Gamma))$ of Γ-connected, strict partial $\mathfrak{G}$-partitions does not in general form a geometric lattice. It is closed under join, hence is a lattice and submodular, but it is not atomic unless Γ is a union of disjoint complete graphs. We show this by demonstrating it for the case $\Gamma = P_3$, the path $v_1v_2v_3$ on three vertices. Let $\gamma = \{\gamma_V\}$ where $V = \{v_1, v_2, v_3\}$ and γ_V is the homogeneous V-tuple corresponding to $\gamma_V(v_1) = \gamma_V(v_2) = 1$ and $\gamma_V(v_3) = g \neq 1$. (We assume $|\mathfrak{G}| \geq 2$.) The $\mathfrak{G}$-partitions $\leq \gamma$ are $o_V = \gamma \wedge 0_V$, $\gamma \wedge \{v_1v_2, v_3\}$ (which is not strict), $\gamma \wedge \{v_1v_3, v_2\}$ (which is not Γ-connected), and $\gamma \wedge \{v_1, v_2v_3\}$. Thus γ has rank two in Q but is not a join of atoms. If Γ is not P_3 but contains it as an induced subgraph, then $o_{v_1v_2v_3}$ and γ lie in Q and their interval in Q, whose only atom is $\gamma \wedge \{v_1, v_2v_3\}$, is not atomic.

The correct Γ-analog of strict partial $\mathfrak{G}$-partitions seems to be the set of strongly Γ-linked partial $\mathfrak{G}$-partitions. The appropriate natural injections $i_{\mathbf{I}''}(\Gamma)$ have adjoints given by

$$i^{\Delta}_{\mathbf{I}''(\Gamma)}(\gamma) = \gamma \wedge \pi(\Gamma\langle \pi_0(\gamma)\rangle),$$
$$i^{\Delta}_{\mathbf{I}''(\Gamma)}(\pi) = \pi \wedge \pi(\Gamma).$$

The former is γ modified by dividing each block into the maximal subsets connected by the Γ-edges between different subblocks of that block.

Theorem 8A. *Let Γ be a simple graph on vertex set V of possibly infinite order $n > 0$. Let $\mathfrak{G}$ be a group of order $M \geq 2$.*

(I) The set of strongly Γ-linked partial $\mathfrak{G}$-partitions of V is the rank n geometric lattice $P^\dagger(\mathfrak{G}, \mathbf{I}''(\Gamma))$. Its meet is $\gamma_1 \wedge''_\Gamma \gamma_2 = i^{\Delta}_{\mathbf{I}''(\Gamma)}(\gamma_1 \wedge \gamma_2)$.

The set of strongly Γ-linked $\mathfrak{G}$-partitions is $P(\mathfrak{G}, \mathbf{I}''(\Gamma))$, whose maximal elements have corank $c(\Gamma)$.

(II) Let $\gamma \in P^\dagger(\mathfrak{G}, \mathbf{I}''(\Gamma))$. Define $\mathbf{I}^* : \pi(\gamma) \times \pi(\gamma) \to \mathcal{P}(\mathfrak{G})$ by setting $H_{AB} = \{\gamma_A(v)\gamma_B(w)^{-1} : v \in A,\ w \in B,\ vw \in E(\Gamma)\}$ and

$$\mathbf{I}^*_{AB} = \begin{cases} \mathfrak{G} & \text{if } |H_{AB}| \geq 2, \\ \mathfrak{G} \backslash \{g\} & \text{if } H_{AB} = \{g\}, \\ \varnothing & \text{if } H_{AB} = \varnothing. \end{cases}$$

Then $[\gamma, 1] \cong P^\dagger(\mathfrak{G}, \mathbf{I}^*)$.

(III) Assume n and M are finite. The characteristic polynomial of $P(\mathfrak{G}, \mathbf{I}''(\Gamma))$ equals $y^{-c(\Gamma)} M^{n-c(\Gamma)} w_\Gamma(1/M, y/M)$.

The characteristic polynomial of $P^\dagger(\mathfrak{G}, \mathbf{I}''(\Gamma))$ is equal to

$$M^n w_\Gamma\left(\frac{1}{M}, \frac{y-1}{M}\right)$$

and its Möbius invariant is $M^n w_\Gamma(1/M, -1/M)$.

Proof. (I) Let $\mathbf{I} = \mathbf{I}''(\Gamma)$. For any partial $\mathfrak{G}$-partition γ, $\Delta(\gamma_B) = \Gamma\langle \pi(\gamma_B)\rangle$, which is connected for all blocks precisely when γ is strongly Γ-linked. The rest of part (I) is immediate from Theorem I.3.1A, observing that $\Delta_{\mathbf{I}} = \Gamma$.

(II) The value $\mathbf{I}^*_{AB}$ from Theorem I.3.1A is the union of $\mathfrak{G} \backslash \{\gamma_A(v)\gamma_B(w)^{-1}\}$ over edges $vw \in E(\Gamma)$. Part (II) follows easily.

(III) Here we have to evaluate $\chi^*_\Phi(y)$ where $\Phi = (\mathfrak{G}\backslash\{1\})\Gamma$. Let $\Gamma_1 = \varnothing$ and $\Gamma_2 = \Gamma$ in formula (I.4.2). Thus

$$\begin{aligned}\chi^*_\Phi(y) &= \sum_{\pi\in\Pi(\Gamma)} M^{|\pi|}\chi_{\Gamma/\pi}\left(\frac{y}{M}\right) = M^n \sum_{\pi\in\Pi(\Gamma)} M^{-\operatorname{rk}\pi}\chi_{\Gamma/\pi}\left(\frac{y}{M}\right) \\ &= M^n w_\Gamma\left(\frac{1}{M}, \frac{y}{M}\right).\end{aligned}$$

(The rank of π is taken in $\Pi(\Gamma)$, where it is $n - |\pi|$.) The rest of the calculations are routine ones from Section I.4. $\square$

We can obtain a formula for the Whitney number polynomial of the underlying gain graph Φ by Proposition I.4.4 together with the evaluation

$$w^*_{\mathbb{B}}(x, M) = \sum_{\tau\in\Pi_B} (M)_{|\tau|} x^{\operatorname{rk}(\mathbb{B})\langle\tau\rangle}. \tag{2}$$

To prove (2) we simply count proper zero-free 1-colorings λ of $\Phi{:}B$ according to the partition τ of B into sets with constant image under λ. The improper edge set of λ consists of the edges of $(\Gamma{:}B)\langle\tau\rangle$ with balanced gains.

Theorem 8B. *Let Γ, $\mathfrak{G}$, etc. be as in part A.*

(I) The set of strongly Γ-linked $\mathfrak{G}$-partitions and ordinary Γ-connected partitions is the geometric lattice $P^{\S}(\mathfrak{G}, \mathbf{I}''(\Gamma))$, of rank $n + 1 - c(\Gamma)$. The top element is $\pi(\Gamma)$. The meet of elements of different types is $\gamma \wedge''_\Gamma \pi = i^{\Delta}_{\mathbf{I}''(\Gamma)}(\gamma \wedge' \pi)$. That of two partitions is as in $\Pi(\Gamma)$.

(II) Let $\gamma \le \pi$ in $P^{\S}(\mathfrak{G}, \mathbf{I}''(\Gamma))$. The interval $[\gamma, \pi]$ is isomorphic to

$$\left(\bigtimes_{C\in\pi} P(\mathfrak{G}, \mathbf{I}^*|\pi(\gamma|_C))\right) \cup \left(\bigtimes_{C\in\pi} \Pi(\Gamma/\pi(\gamma|_C))\right),$$

where $\mathbf{I}^*$ is defined in part A.

(III) Let n and M be finite. The characteristic polynomial of $P^{\S}(\mathfrak{G}, \mathbf{I}''(\Gamma))$ is

$$\frac{y-1}{y^{c(\Gamma)}} M^{n-c(\Gamma)} w_\Gamma\left(\frac{1}{M}, \frac{y}{M}\right).$$

Its Möbius invariant equals $-\sum_0^r M^{r-i} w_{ir}(\Pi(\Gamma))$, where $r = n - c(\Gamma) = \operatorname{rk}\Pi(\Gamma)$.

Proof. These parts are all standard by the methods of Sections I.3 and I.4. $\square$

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