# FACES OF A HYPERPLANE ARRANGEMENT ENUMERATED BY IDEAL DIMENSION, WITH APPLICATION TO PLANE, PLAIDS, AND SHI

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ABSTRACT. The *ideal dimension* of a real affine set is the dimension of the intersection of its projective topological closure with the infinite hyperplane. We obtain a formula for the number of faces of a real hyperplane arrangement having given dimension and ideal dimension. We apply the formula to the plane, to plaids, which are arrangements of parallel families in general position, and to affinographic arrangements. We compare two definitions of ideal dimension and ask about a complex analog of the enumeration.

## 1. INTRODUCTION

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An arrangement of hyperplanes in real affine space $\mathbb{E}^d$ is a finite set $\mathcal{H}$ of hyperplanes (affine subspaces with codimension 1), together with the dissection of the affine space into relatively open cells of various dimensions (the *faces* of $\mathcal{H}$; the $d$-dimensional faces are called *regions*) that is implied by the way the hyperplanes divide the space in halves. Let $\mathcal{L}$ be the set of nonvoid flats formed by intersecting hyperplanes of $\mathcal{H}$, including $\mathbb{E}^d$ as the intersection of no hyperplanes. Ordered by reverse inclusion, this is a geometric semilattice [11]; accordingly it has a characteristic polynomial $p_{\mathcal{L}}(\lambda)$. A well known result is that the number of regions equals $|p_{\mathcal{L}}(\lambda)|$ [12]. Similar formulae express the numbers of faces of each dimension.

Richard Ehrenborg, in an early version of [4], proposed a refinement of the problem of enumerating faces. Each face $F$ has an infinite part, where its projective topological closure $\mathrm{cl}_{\mathbb{P}}(F)$ intersects the infinite hyperplane $h_\infty$ (the complement of $\mathbb{E}^d$ in the real projective space $\mathbb{P}^d$) in a set $\mathrm{cl}_{\mathbb{P}}(F) \cap h_\infty$ whose relative interior is a face (possibly empty; in a projective arrangement $\mathcal{A}$ the empty set is a face when $\bigcap \mathcal{A} = \emptyset$) in the dissection of $h_\infty$ induced by $\mathcal{H}$. This face is $[F]_\infty$, the *ideal part* of $F$. We call the dimension of $[F]_\infty$ the *ideal dimension* of $F$, written $\dim_\infty F$. Ehrenborg's problem is to count the number of faces having a given dimension and ideal dimension. (Ehrenborg used a slightly different concept, the *level* of $F$, which for a face of an arrangement equals $\dim_\infty F + 1$. See Section 7.) In this article we present a general formula for the solution. Our formula does not directly yield Ehrenborg's subtle expression for the numbers associated with the Shi arrangement (which consists of all hyperplanes with equations $x_j - x_i = 0$ or 1 for $1 \leq i < j \leq d$) and extensions. Nonetheless it seems worthy of promulgation. We shall, after introductory definitions, present the general formula; apply it to examples including planar arrangements, arrangements of a simple type recently dubbed a *plaid* by Alexanderson and Wetzel [3] but introduced long ago—for dimensions 2 and 3—by Steiner [10], and affinographic arrangements such as the extended Shi and Linial arrangements; discuss the equivalence and inequivalence of level and ideal

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dimension; and conclude with the open problem of finding an analog of ideal dimension for complex hyperplane arrangements.

## 2. Background and definitions

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We work in both real affine $d$-space $\mathbb{E}^d$ and real projective space $\mathbb{P}^d$, whose ideal hyperplane (the hyperplane at infinity) is $h_\infty \cong \mathbb{P}^{d-1}$. The affine span of an affine set is $\operatorname{aff} X$; the projective span of any set is $\operatorname{span} X$. In particular, the projective span of an affine flat $s$ is $s_\mathbb{P} = \operatorname{span} s = \operatorname{cl}_\mathbb{P} s$.

Given an arrangement of hyperplanes, $\mathcal{H}$, in $\mathbb{E}^d$, its *intersection semilattice*,

$$\mathcal{L}(\mathcal{H}) = \{\bigcap \mathcal{S} : \mathcal{S} \subseteq \mathcal{H}, \bigcap \mathcal{S} \neq \emptyset\},$$

is ordered by reverse inclusion. The intersection lattice of a projective arrangement $\mathcal{A}$ is

$$\mathcal{L}(\mathcal{A}) = \{\bigcap \mathcal{S} : \mathcal{S} \subseteq \mathcal{A}\},$$

that is, the null set is not excluded. The *projectivization* of $\mathcal{H}$ is

$$\mathcal{H}_\mathbb{P} = \{h_\mathbb{P} : h \in \mathcal{H}\} \cup \{h_\infty\}.$$

Since the intersection (semi)lattice of an arrangement is ordered by reverse inclusion, its minimum element is the ambient space, called $\hat{0}$. The maximum element of $\mathcal{L}(\mathcal{H}_\mathbb{P})$, as of $\mathcal{L}(\mathcal{H}^\infty)$, is the ideal flat $1_\infty = \bigcap \mathcal{H}_\mathbb{P}$.

For any arrangement $\mathcal{A}$, affine or projective, the *induced arrangement* in a flat $t \in \mathcal{L}(\mathcal{A})$ is

$$\mathcal{A}^t = \{h \cap t : h \in \mathcal{A}, h \not\supseteq t\};$$

the induced arrangement at infinity of an affine arrangement $\mathcal{H}$ is

$$\mathcal{H}^\infty = (\mathcal{H}_\mathbb{P})^{h_\infty}.$$

Notice that $\bigcap \mathcal{H}^\infty = 1_\infty$. The *subarrangement of* $\mathcal{A}$ *over* any affine or projective set $X$ is

$$\mathcal{A}(X) = \{h \in \mathcal{H} : h_\mathbb{P} \supseteq X\},$$

where in the projective case $h_\mathbb{P}$ is $h$. Usually $X$ is an affine or projective (possibly ideal) flat of $\mathcal{A}$. The corresponding intersection semilattices are

$$\mathcal{L}(\mathcal{A}^t) = \{t' \in \mathcal{L}(\mathcal{A}) : t' \geq t\},$$
$$\mathcal{L}(\mathcal{A}(t)) = \{t' \in \mathcal{L}(\mathcal{A}) : t' \leq t\}$$

if $t \in \mathcal{L}(\mathcal{A})$, and for an affine arrangement

$$\mathcal{L}(\mathcal{H}_\mathbb{P}^s) = \{s' \in \mathcal{L}(\mathcal{H}_\mathbb{P}) : s' \geq s\},$$
$$\mathcal{L}(\mathcal{H}_\mathbb{P}(s)) = \{s' \in \mathcal{L}(\mathcal{H}_\mathbb{P}) : s' \leq s\},$$
$$\mathcal{L}(\mathcal{H}(s)^\infty) = \{s' \in \mathcal{L}(\mathcal{H}^\infty) : s' \leq s\}$$

if $s \in \mathcal{L}(\mathcal{H}_\mathbb{P})$.

The infinite or *ideal part* of a set $X$, affine or projective, is $[X]_\infty = \operatorname{cl}_\mathbb{P}(X) \cap h_\infty$. The *ideal dimension* of $X$ is $\dim_\infty X = \dim [X]_\infty$. The *rank* of $X$ is $\operatorname{rk} X = \operatorname{codim} X = d - \dim X$. Its *ideal rank* is $\operatorname{rk}_\infty X = d - 1 - \dim_\infty X$. In particular, for an ideal set $X$, $\dim_\infty X = \dim X$ and $\operatorname{rk}_\infty X = d - 1 - \dim X = \operatorname{rk} X - 1$.

Now let $\mathcal{A}$ be an arrangement in $\mathbb{E}^d$ or $\mathbb{P}^{d-1}$. The rank of $\mathcal{A}$ is $\operatorname{rk}(\mathcal{A}) = \max\{\operatorname{rk} t : t \in \mathcal{L}(\mathcal{A})\}$. The *characteristic polynomial* of $\mathcal{A}$ is

$$p_{\mathcal{A}}(\lambda) = \sum_{t \in \mathcal{L}(\mathcal{A})} \mu(\hat{0}, t)\lambda^{\delta + \dim t}$$

where $\delta$ is 0 for an affine and 1 for a projective arrangement and $\mu$ is the combinatorial Möbius function. (An appropriate reference for the definition of $\mu$ is [12].) The *Whitney-number polynomial*[1] is

$$w_{\mathcal{A}}(x, \lambda) = \sum_{s \in \mathcal{L}(\mathcal{A})} x^{\operatorname{rk} s} p_{\mathcal{A}^s}(\lambda).$$

Let

$$\tilde{r}(\mathcal{A}) = (-1)^d p_{\mathcal{A}}(-1) \qquad \text{and} \qquad b(\mathcal{A}) = (-1)^{\operatorname{rk}(\mathcal{A})} p_{\mathcal{A}}(1)$$

and define $\tilde{f}_k(\mathcal{A})$ and $b_k(\mathcal{A})$ by the generating functions

$$\sum f_k(\mathcal{A}) x^{d-k} = (-1)^d w_{\mathcal{A}}(-x, -1) \qquad \text{and} \qquad \sum b_k(\mathcal{A}) x^{d-k} = (-1)^{\operatorname{rk}(\mathcal{A})} w_{\mathcal{A}}(-x, 1).$$

Then for a real affine arrangement $\mathcal{H}$,

$$\text{(2.1)} \qquad \{\text{Deqa}\} \qquad \begin{aligned} \tilde{r}(\mathcal{H}) &= f_d(\mathcal{H}) = r(\mathcal{H}), \text{ the number of regions,} \\ b(\mathcal{H}) &= b_d(\mathcal{H}) = \text{ the number of relatively bounded regions,} \\ \tilde{f}_k(\mathcal{H}) &= f_k(\mathcal{H}), \text{ the number of faces of dimension } k, \text{ and} \\ b_k(\mathcal{H}) &= \text{ the number of relatively bounded faces of dimension } k, \end{aligned}$$

by [12, Theorems A and C]. (A face is called *relatively bounded* if, whenever it contains a ray, it contains the whole line generated by the ray; this is equivalent to the definition in [12, §2C, p. 25].) For an arrangement $\mathcal{A}$ in $\mathbb{P}^{d-1}$,

$$\text{(2.2)} \qquad \{\text{Deqp}\}\ \tfrac{1}{2}\tilde{f}_k(\mathcal{A}) = f_k(\mathcal{A}), \text{ the number of } k\text{-faces,}$$

except that when $k = d - \operatorname{rk}\mathcal{A}$, $\tilde{f}_k(\mathcal{A}) = f_k(\mathcal{A}) = 1$ [12, Theorem B].

## 3. The general formulae

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How do we find the number of faces of $\mathcal{H}$ having dimension $k$ and ideal dimension $l$? Call this number $f_{kl}$. (Note that $f_{kl} = 0$ unless $d - \operatorname{rk}\mathcal{H} - 1 \leq l < k \leq d$.)

**Theorem 3.1.** *{Genum} Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{E}^d$. The ordinary generating function of faces by rank and ideal rank is*

$$\sum\sum_{-1 \leq l < k \leq d} f_{kl} x^{d-k} z^l = (-1)^d \sum_{s \in \mathcal{L}(\mathcal{H}^\infty)} w_{\mathcal{H}(s)}(-x, 1) p_{\mathcal{H}^s}(-1) z^{\dim s}.$$

We shall work up to the proof by stages. In all but the last step we deal with regions. The first step analyzes the regions with ideal part equal to a specific ideal face $f$.

**Proposition 3.2.** *{Gface} Let $R$ be a region of $\mathcal{H}$, $f = [R]_\infty$, $s = \operatorname{span} f$, and $Q$ be the region of $\mathcal{H}(s)$ that contains $R$.*

(1) *$Q$ is a relatively bounded region of $\mathcal{H}(s)$.*

(2) *The following are mutually exclusive and exhaustive alternatives:*

[1] Called the "Möbius polynomial" in [12]; then the "Whitney polynomial" but that name has been used to mean another polynomial.

(a) $\mathcal{H} = \mathcal{H}(s)$, $R = Q$, $f = s = 1_\infty$, *and $R$ is the only region $R'$ of $\mathcal{H}$ such that $R' \subseteq Q$ and $[R']_\infty = f$.*
(b) $\mathcal{H} \supset \mathcal{H}(s)$, $R \subset Q$, $f \subset s \neq 1_\infty$, *and there exist precisely two regions $R'$ of $\mathcal{H}$ such that $R' \subseteq Q$ and $[R']_\infty = f$.*

Part of the proof is contained in a lemma.

**Lemma 3.3.** {*Grb*} *Let $F$ be a face of $\mathcal{H}$. Then $[F]_\infty = 1_\infty \iff F$ is relatively bounded in $\mathcal{H}$.*

*Proof.* What we must prove is that $[F]_\infty \not\supset 1_\infty$. It is clear that $[F]_\infty \supset 1_\infty \iff$ there is a point $p \in [F]_\infty \setminus 1_\infty \iff$ (*) there is a ray $r$ in $F$ (i.e., that in the direction $p$) whose line $\operatorname{aff}(r)$ does not meet $1_\infty$.

Now, suppose (*) holds true. There is a hyperplane $h \in \mathcal{H}$ such that $p \notin h$. Consequently $h$ intersects $\operatorname{aff}(r)$, so $\operatorname{aff}(r) \not\subseteq F$. Therefore $F$ is not relatively bounded.

Conversely, suppose $F$ is not relatively bounded. Then there are a ray $r \subseteq F$ and a hyperplane $h \in \mathcal{H}$ that intersects $\operatorname{aff}(r)$. Evidently $[\operatorname{aff}(r)]_\infty \notin 1_\infty$.

Thus (*) is true if and only if $F$ is not relatively bounded. □

*Proof of Proposition 3.2.* Part (1) follows from Lemma 3.3. For part (2) there are two cases: $\mathcal{H} = \mathcal{H}(f)$ and $\mathcal{H} \neq \mathcal{H}(f)$. (Recall that $\mathcal{H}(f) = \mathcal{H}(s)$ virtually by definition.) The former is trivial; therefore we assume that $\mathcal{H} \supset \mathcal{H}(f)$.

In a small ball around a point $p \in F$, $Q$ looks like two pieces, $q_1$ and $q_2$, diametrically opposite across $f$. If we take any $h \in \mathcal{H} \setminus \mathcal{H}(s)$, then $h$ separates $q_1$ from $q_2$, in that a path connecting them in $Q$ must pass through $h$. Thus $h$ divides $Q$ into two parts, $Q_1^h \supseteq q_1$ and $Q_2^h \supseteq q_2$, both of which have ideal part containing $f$.

Now consider how $Q$ is broken up by $\mathcal{H}$. It may have many pieces, but none contains both $q_1$ and $q_2$. The parts $Q_1$ and $Q_2$ that do contain $q_1$ and $q_2$ have ideal parts containing $f$. Since the ideal part of a face of $\mathcal{H}$ is a face of $\mathcal{H}^\infty$, $[Q_1]_\infty = [Q_2]_\infty = f$; but any other region contained in $Q$ has a different ideal part since it contains neither $q_1$ nor $q_2$. Thus there are just two regions $R'$ of $\mathcal{H}$ as stated in part (b), namely $Q_1$ and $Q_2$. □

**Corollary 3.4.** {*Genumf*} *Let $f$ be an ideal face of $\mathcal{H}$. The number of regions of $\mathcal{H}$ whose ideal part is $f$ equals $2b(\mathcal{H}(f))$ if $f \neq 1_\infty$, but $b(\mathcal{H})$ if $f = 1_\infty$.*

*Proof.* Sum Proposition 3.2 over all relatively bounded regions $Q$ in $\mathcal{H}(f)$. □

**Corollary 3.5.** {*Genums*} *Let $s \in \mathcal{L}(\mathcal{H}^\infty)$. The number of regions of $\mathcal{H}$ whose ideal part spans $s$ equals $2b(\mathcal{H}(s))r(\mathcal{H}^s)$ if $s \neq 1_\infty$, but $b(\mathcal{H})$ if $s = 1_\infty$.*

*Proof.* Sum Corollary 3.4 over all regions $f$ of $\mathcal{H}^s$. □

*Proof of Theorem 3.1.* We sum Corollary 3.5 over all ideal flats $s$. Let $l$ be any dimension in the range $\dim 1_\infty < l < d$. Then

$$\begin{aligned} f_{dl} &= \sum_{\substack{s \in \mathcal{L}(\mathcal{H}^\infty) \\ \dim s = l}} b(\mathcal{H}(s)) \cdot 2r(\mathcal{H}^s) \\ &= \sum_{\substack{s \in \mathcal{L}(\mathcal{H}^\infty) \\ \dim s = l}} (-1)^{\operatorname{rk}\mathcal{H}(s)} p_{\mathcal{H}(s)}(1) \cdot (-1)^{1+\dim s} p_{\mathcal{H}^s}(-1) \end{aligned}$$

by (2.1) and (2.2). When $l = \dim 1_\infty$, we have $s = 1_\infty$, $\mathcal{H}(s) = \mathcal{H}$, and $\mathcal{H}^s = \emptyset$, so $p_{\mathcal{H}^s}(\lambda) = \lambda^{1+\dim s}$; therefore

$$f_{dl} = b(\mathcal{H}) = (-1)^{\operatorname{rk}\mathcal{H}(s)} p_{\mathcal{H}(s)}(1) \cdot (-1)^{1+\dim s} p_{\mathcal{H}^s}(-1),$$

just as for the other flats. Summing over $l$ and recalling that $\operatorname{rk}\mathcal{H}(s) = d - \dim s - 1$, we have

$$\{\text{Geqr}\} \sum_l f_{dl} z^l = (-1)^d \sum_{s \in \mathcal{L}(\mathcal{H}^\infty)} p_{\mathcal{H}(s)}(1) p_{\mathcal{H}^s}(-1) z^{\dim s}. \tag{3.1}$$

To get the formula for $k$-faces of $\mathcal{H}$ we look into each flat $t \in \mathcal{L}(\mathcal{H})$ to count the regions of $\mathcal{H}^t$ by ideal dimension. In $\mathcal{H}^t$ we have again Equation (3.1) but with $\mathcal{H}$ replaced by $\mathcal{H}^t$; thus $\mathcal{H}(s)$ replaced by $\mathcal{H}^t(s) = \mathcal{H}(s)^t$ (for $s \in \mathcal{L}((\mathcal{H}^t)^\infty)$, or equivalently $t < s \in \mathcal{L}(\mathcal{H}^\infty)$); $\mathcal{H}(s)$ by $(\mathcal{H}^t)^s = \mathcal{H}^s$; and $d$ by $k = \dim t$. Summing over all $t \in \mathcal{L}(\mathcal{H})$ gives

$$\begin{aligned}
\sum_{-1 \le l < k \le d}\sum f_{kl} x^{d-k} z^l &= \sum_{t \in \mathcal{L}(\mathcal{H})} (-1)^{\dim t} x^{d-\dim t} \sum_{t < s \in \mathcal{L}(\mathcal{H}^\infty)} p_{\mathcal{H}(s)^t}(1) p_{\mathcal{H}^s}(-1) z^{\dim s} \\
&= (-1)^d \sum_{s \in \mathcal{L}(\mathcal{H}^\infty)} \left\{ \sum_{t \in \mathcal{L}(\mathcal{H}(s))} (-x)^{\operatorname{rk} t} p_{\mathcal{H}(s)^t}(1) \right\} p_{\mathcal{H}^s}(-1) z^{\dim s} \\
&= (-1)^d \sum_{s \in \mathcal{L}(\mathcal{H}^\infty)} w_{\mathcal{H}(s)}(-x, 1) p_{\mathcal{H}^s}(-1) z^{\dim s},
\end{aligned}$$

as in the statement of the theorem. □

In particular, the generating function for regions by ideal dimension is given by Equation (3.1).

## 4. In the plane

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In $\mathbb{E}^2$ it is easy to calculate the face numbers directly. This provides a check on the general formula and a relatively simple example to show how to use Theorem 3.1.

In an affine planar arrangement $\mathcal{H}$ (which we assume to have rank 2), let $v_i$ be the number of points at which $i$ lines intersect (for $i \geq 2$) and $c_j$ be the number of parallel pencils of $j$ lines (for $j \geq 1$). Let $I$ be the number of incident point-line pairs; thus $I = \sum i v_i$. Let $v = \sum v_i$, the number of points, and $c = \sum c_j$, the number of parallel pencils (including pencils of only one line). Finally, let $n$ be the number of lines, so $n = \sum j c_j$. The numbers of bounded faces are, from [12, §5E, p. 612],

$$f_{2,-1} = I - v - n + 1, \qquad f_{1,-1} = I - n, \qquad f_{0,-1} = v.$$

The numbers of faces with ideal dimension 0 are easily seen to be

$$f_{2,0} = 2(n-c), \qquad f_{1,0} = 2n.$$

The number of faces with the largest ideal dimension is

$$f_{2,1} = 2c.$$

Now we evaluate the general formula when $d = 2$. In the calculation we retain the variable in $p_{\mathcal{H}^s}$ as $\lambda = -y$ for reference in Section 8; then at the end set $y = 1$.

The ideal flats $s$ are of three kinds: $s = h_\infty$, $s =$ a point $p_k$ which is the ideal point of the $k$-th parallel pencil, and $s = 1_\infty$.

*Case 1.* If $s = h_\infty$, then $\mathcal{H}(s) = \emptyset$ in $\mathbb{E}^2$ and $\mathcal{H}^s$ is an arrangement of $c$ points in the projective line. Thus

$$w_{\mathcal{H}(s)}(x,\lambda) = \lambda^2 \qquad \text{and} \qquad p_{\mathcal{H}^s}(\lambda) = (\lambda-1)(\lambda-c+1),$$

so the term for this case is

$$\{\text{2line}\}\ w_{\mathcal{H}(s)}(-x,1)p_{\mathcal{H}^s}(-y)z^{\dim s} = 1\cdot(y+1)(y+c-1)\cdot z. \tag{4.1}$$

*Case 2.* If $s = p_k$, then $\mathcal{H}(s)$ consists of $j_k$ parallel lines and $\mathcal{H}^s$ is an arrangement of one hyperplane (the null set) in $p_k \cong \mathbb{P}^0$. Therefore $p_{\mathcal{H}^s}(\lambda) = \lambda - 1$ and

$$\begin{aligned} w_{\mathcal{H}(s)}(x,\lambda) &= p_{\mathcal{H}(s)}(\lambda) + j_k p_{\mathcal{H}(s)^l}(\lambda)x \\ &= (\lambda^2 - j_k\lambda) + j_k\lambda x, \end{aligned}$$

where $l$ denotes any of the lines in the pencil and $\mathcal{H}(s)^l = \emptyset$ in $l \cong \mathbb{E}^1$. Summing over all pencils, the terms for pencils total

$$\{\text{2point}\}\ \begin{aligned} \sum_s w_{\mathcal{H}(s)}(-x,1)p_{\mathcal{H}^s}(-y)z^{\dim s} &= \sum_{\text{pencils}} [1-j_k-j_kx][-y-1]x^0 \\ &= (nx+n-c)(y+1)z^0. \end{aligned} \tag{4.2}$$

*Case 3.* If $s = 1_\infty$, then $\mathcal{H}(s) = \mathcal{H}$ and $\mathcal{H}^s$ is the null arrangement in $\mathbb{P}^{-1}$. Thus $p_{\mathcal{H}^s}(\lambda) = 1$ and

$$w_{\mathcal{H}(s)}(x,\lambda) = w_{\mathcal{H}}(x,\lambda) = \sum_{t\in\mathcal{L}(\mathcal{H})} x^{\operatorname{rk} t} p_{\mathcal{H}^t}(\lambda).$$

To evaluate this Whitney-number polynomial we examine each dimension of $t$. If $t = \mathbb{E}^2$, then $\mathcal{H}^t = \mathcal{H}$ so

$$\begin{aligned} p_{\mathcal{H}^t}(\lambda) &= \lambda^2 - n\lambda + \sum_P [\#(\text{lines on } P) - 1] \\ &= \lambda^2 - n\lambda + I - v, \end{aligned}$$

the sum being over all vertices of $\mathcal{H}$. If $t$ is a line $l$, then $\mathcal{H}^t$ is an arrangement of points on $l$ so $p_{\mathcal{H}^t}(\lambda) = \lambda - \#(\text{points on } l)$; therefore

$$\sum_l p_{\mathcal{H}^t}(\lambda) = n\lambda - I.$$

Taking $t$ a vertex $P$, $\mathcal{H}^t =$ the null arrangement in $P \cong \mathbb{E}^0$ so $p_{\mathcal{H}^t}(\lambda) = 1$; therefore

$$\sum_P p_{\mathcal{H}^t}(\lambda) = v.$$

So we have in Case 3

$$\{\text{2null}\}\ w_{\mathcal{H}(s)}(-x,1)p_{\mathcal{H}^s}(-y)z^{\dim s} = [(1-n+I-v) - x(n-I) + x^2v]\cdot 1\cdot z^{-1}. \tag{4.3}$$

Summing (4.1–4.3) yields

$$\{\text{2refined}\}\ z(y+1)(y+c-1) + z^0(y+1)(nx+n-c) + z^{-1}[(1-n+I-v)+x(n-I)+x^2v]. \tag{4.4}$$

Setting $y = 1$ gives the face enumerator by dimension and ideal dimension:

$$z(2c) + x(2n) + 2(n-c) + z^{-1}(I-v-n+1) + z^{-1}x(I-n) + z^{-1}x^2v.$$

This agrees exactly with the direct enumeration.

## 5. Plaids

{P} *Date added: original; heavily corrected January 2001.*

A plaid $\mathcal{P} = \mathcal{P}_d(m_1, \ldots, m_n)$ starts with an arrangement $\mathcal{H}_{n,d}$ of $n$ hyperplanes in general position in $\mathbb{E}^d$; then the $i$-th hyperplane is replaced by $m_i$ parallel copies of itself in general position with respect to the other hyperplanes. Plaids, after Steiner's old work [10], were taken up in modern times by Alexanderson and Wetzel [2, 3] and, inspired by them, by me in [13]. To avoid trivialities we assume $\mathcal{P}$ has rank $d$ (that is, $n \geq d$). $\mathcal{L} = \mathcal{L}(\mathcal{P})$ is the affine intersection semilattice. Write $\mathbf{m} = (m_1, \ldots, m_n)$ and $[n]^{(k)} =$ the class of $k$-element subsets of $[n] = \{1, 2, \ldots, n\}$.

Now, $[\mathcal{P}]_\infty$ consists of $n$ projective hyperplanes in general position in $d-1$ dimensions. The projective intersection lattice is therefore $\operatorname{Lat} U(n,d)$, the lattice of the uniform matroid of $n$ points in rank $d$. Its characteristic polynomial is

$$p_d(n;\lambda) = \sum_{i=0}^{d-1} (-1)^i [\lambda^{d-i} - 1] \binom{n}{i}.$$

The arrangement $\mathcal{P}^s$ induced by $\mathcal{P}$ in an ideal flat $s$ with $\operatorname{rk}_\infty s = j$ evidently has lattice $\operatorname{Lat} U(n-j, d-j)$. The number of regions of $\mathcal{P}^s$ is therefore

$$r(\mathcal{P}^s) = (-1)^{d-j} p_{d-j}(n-j;-1) = \sum_{l=0}^{d-j-1} [1 - (-1)^{d-j-l}] \binom{n-j}{l}.$$

The affine intersection semilattice $\mathcal{L}(\mathcal{P})$ starts with $\mathcal{L}(\mathcal{H}_{n,d}) \cong [\operatorname{Lat} F(n)]_0^d$, which means ranks 0 through $d$ of the lattice of the free matroid on $n$ points. An intersection of $r$ original hyperplanes $h_{i_1}, h_{i_2}, \ldots, h_{i_r}$ expands to $m_{i_1} m_{i_2} \cdots m_{i_r}$ intersections in the plaid, each of which we say has *type* $\{i_1, i_2, \ldots, i_r\}$. Thus if $s_r(\mathbf{m})$ denotes the $r$-th elementary symmetric function of the $m_i$'s, the plaid has $s_r(\mathbf{m})$ intersections $t$ of rank $r$. Since $t$ is a simple intersection, $\mu(\hat{0}, t) = (-1)^r$. Thus the characteristic polynomial is

$$p_{\mathcal{P}}(\lambda) = p_d(\mathbf{m};\lambda) = \sum_{r=0}^{d} (-1)^r s_r(\mathbf{m}) \lambda^{d-r}.$$

To calculate $w_{\mathcal{P}}(x,\lambda)$ we need the characteristic polynomials of all $\mathcal{P}^t$ for $t \in \mathcal{L}(\mathcal{P})$. Suppose $t$ has type $S$ and $\dim t = k$, so that $|S| = d-k$. Then $\mathcal{P}^t \cong \mathcal{P}_k(\mathbf{m}|_{S^c})$, where $S^c = [n] \setminus S$ and $\mathbf{m}|_T$ is $\mathbf{m}$ restricted to elements indexed by $T$. Thus $p_{\mathcal{P}^t}(\lambda) = p_k(\mathbf{m}|_{S^c};\lambda)$. The number of $t$ of type $S$ is $\prod_{i \in S} m_i$. So we have:

$$\begin{aligned} w_{\mathcal{P}}(x,\lambda) &= \sum_{k=0}^{d} \sum_{\substack{S \subseteq [n] \\ |S| = d-k}} x^{d-k} \left( \prod_{i \in S} m_i \right) \sum_{l=0}^{k} (-1)^l s_l(\mathbf{m}|_{S^c}) \lambda^{k-l} \\ &= \sum_{0 \leq l \leq k \leq d} x^{d-k} (-1)^l \lambda^{k-l} \sum_{|S| = d-k} \left( \prod_{i \in S} m_i \right) \sum_{\substack{T \subseteq S^c \\ |T| = l}} \left( \prod_{i \in T} m_i \right) \end{aligned}$$

$$= \sum_{r=0}^{d} \sum_{l=0}^{r} x^{r-l}(-1)^l \lambda^{d-r} \binom{r}{l} s_r(\mathbf{m})$$

where $r = d - k + l$,

$$= \sum_{r=0}^{d} \lambda^{d-r}(x-1)^r s_r(\mathbf{m}).$$

Thus

$$(-1)^d w_{\mathcal{P}}(-x, 1) = \sum_{r=0}^{d} (-1)^{d-r}(x+1)^r s_r(\mathbf{m}),$$

from which by extracting the coefficient of $x^{d-k}$ and setting $l = d - r$ we find the number of bounded $k$-faces,

$$\{\text{Pbd}\}\ b_k(\mathcal{P}) = \sum_{l=0}^{k} (-1)^l \binom{d-l}{d-k} s_{d-l}(\mathbf{m}). \tag{5.1}$$

Now to find out the distribution of unbounded faces by ideal part. Each ideal flat $s \neq 1_\infty$, with ideal dimension $l \geq 0$, is the intersection of $r = d - 1 - l$ hyperplanes of $\mathcal{H}_{n,d}$, which we identify by their index set $S \subseteq [n]$. Evidently, $\mathcal{L}(\mathcal{P}^s) \cong \operatorname{Lat} U(n-r, d-r)$. The restricted arrangement $\mathcal{P}(s)$ is the direct sum of the $r$ 1-dimensional plaids $\mathcal{P}_1(m_i)$ for $i \in S$, each $\mathcal{P}_1(m_i)$ consisting of $m_i$ points in a line $\mathbb{E}^1$. Since $w_{\mathcal{P}_1(m)}(x, \lambda) = \lambda + m(x-1)$, we have

$$w_{\mathcal{P}(s)}(x, \lambda) = \prod_{i \in S} [\lambda + m_i(x-1)].$$

Therefore, the term in Theorem 3.1 that we associate with all $s$ of ideal rank $r$ is

$$\begin{aligned}
(-1)^d & \sum_{\substack{s \in \mathcal{L}(\mathcal{P}^\infty) \\ \operatorname{rk}_\infty s = r}} w_{\mathcal{H}(s)}(-x, 1) p_{\mathcal{H}^s}(-y) z^{\dim s} \\
&= (-1)^d \sum_{\substack{S \subseteq [n] \\ |S| = r}} (-1)^r \prod_{i \in S} \big[1 + m_i(x+1)\big] \cdot p_{d-r}(n-r; -y) z^{d-1-r} \\
&= \sum_{|S|=r} \sum_{q=0}^{r} (x+1)^q \sum_{\substack{T \subseteq S \\ |T| = q}} \prod_{i \in T} m_i \cdot \sum_{j=0}^{d-1-r} \big[y^{d-r-j} - (-1)^{d-r-j}\big] \tbinom{n-r}{j} z^{d-1-r} \\
&= \sum_{q=0}^{r} (x+1)^q s_q(\mathbf{m}) \tbinom{n-q}{r-q} \sum_{j=0}^{d-1-r} \big[y^{d-r-j} - (-1)^{d-r-j}\big] \tbinom{n-r}{j} z^{d-1-r}.
\end{aligned}$$

We sum this over $0 \leq r \leq d-1$, to get the formula on the right-hand side of Theorem 3.1 excluding the term corresponding to $s = 1_\infty$:

$$(-1)^d \sum_{r=0}^{d-1} \sum_{\substack{s \in \mathcal{L}(\mathcal{P}^\infty) \\ \operatorname{rk}_\infty s = r}} w_{\mathcal{H}(s)}(-x, 1) p_{\mathcal{H}^s}(-y) z^{d-1-r}$$

$$= \sum_{r=0}^{d-1} \sum_{q=0}^{r} (x+1)^q s_q(\mathbf{m}) \binom{n-q}{n-r} \sum_{j=0}^{d-1-r} \left[ y^{d-r-j} - (-1)^{d-r-j} \right] \binom{n-r}{j} z^{d-1-r}$$

$$= \sum_{q=0}^{d-1} (x+1)^q s_q(\mathbf{m}) \sum_{l=0}^{d-1-q} \binom{n-q}{n-d+1+l} \sum_{j=0}^{l} \left[ y^{l+1-j} - (-1)^{l+1-j} \right] \binom{n-d+1+l}{j} z^l$$

by setting $r = d-1-l$.

Our first conclusion is the generating function for unbounded $k$-faces by ideal dimension, which we obtain by setting $y = 1$. That is

$$\sum\sum_{0 \le l < k \le d} f_{kl} x^{d-k} z^l = \sum_{q=0}^{d-1} (x+1)^q s_q(\mathbf{m}) \sum_{l=0}^{d-1-q} \binom{n-q}{n-d+1+l} \sum_{j=0}^{l} [1 + (-1)^{l-j}] \binom{n-d+1+l}{j} z^l.$$

We extract from this that

$$\{\text{Punbd}\}\ f_{kl}(\mathcal{P}) = \sum_{q=d-k}^{d-1} \binom{q}{d-k} s_q(\mathbf{m}) \binom{n-q}{n-d+1+l} \sum_{j=0}^{l} \binom{n-d+1+l}{j} [1 + (-1)^{l-j}] \tag{5.2}$$

affine $k$-faces with ideal part of dimension $l$, if $0 \le l < d$. (For ideal part of dimension $-1$ we are counting bounded $k$-faces of $\mathcal{P}$, which have the different formula in Equation (5.1).) Notice the slight simplification by taking into account the parity of $l - j$.

The second conclusion is a refined characteristic polynomial for mention in Section 8. This is obtained by setting $x = 0$ and adding in the number of bounded regions from (5.1). It is

(5.3)

$$\{\text{Prefined}\}\ s_d(\mathbf{m}) + \sum_{q=0}^{d-1} s_q(\mathbf{m}) \left\{ (-1)^{d-q} + \sum_{l=0}^{d-1-q} \binom{n-q}{n-d+1+l} \sum_{j=0}^{l} \left[ y^{\,l+1-j} - (-1)^{l+1-j} \right] \binom{n-d+1+l}{j} z^l \right\}.$$

## 6. Affinographic arrangements

{GG} *Date added: Original; heavily revised and extended January 2001.*

An arrangement $\mathcal{H}$ in $\mathbb{E}^d$ whose hyperplanes have the form $x_j - x_i = \alpha$ (we call such an arrangement *affinographic*) is a representation of a real, additive gain graph. Take a graph on vertex set $[d] = \{1, 2, \ldots, d\}$; to each oriented edge $e$ assign a real number $\varphi(e)$ with the rule that, if $e^{-1}$ is $e$ with orientation reversed, then $\varphi(e^{-1}) = -\varphi(e)$. This constitutes an *additive real gain graph* $\Phi$. The underlying graph is called $\|\Phi\|$. The edge corresponding to the hyperplane $x_j - x_i = \alpha$ is an edge $e$ with endpoints $i$ and $j$ and has gain $\varphi(e; ij) = \alpha$ (in the direction from $i$ to $j$), whence $\varphi(e; ji) = -\alpha$. $\mathcal{H}$ is called the *affinographic representation* of $\Phi$—in full, the *canonical affine hyperplanar lift representation*—because $\mathcal{L}(\mathcal{H})$ is derived from the extended lift matroid $L_0(\Phi)$ [14, §IV.4]. The essential facts for us are that

(1) {GGfact1} The point set of $L_0$ is $E_0 = E(\Phi) \cup \{e_0\}$, $e_0$ being a special element called the 'extra point'. $e_0$ corresponds to $h_\infty$.

(2) {GGfact2} $\mathcal{L}(\mathcal{H}_{\mathbb{P}})$ is isomorphic to the lattice of closed sets of $L_0$. The affine flats correspond to balanced closed sets (those that do not contain $e_0$); the ideal flats correspond to closed sets that ae unbalanced. (See [14, Part II] for the technical terms.)

(3) {GGfact3} $\mathcal{L}(\mathcal{H}^\infty) \cong \Pi(\Phi)$, the lattice of connected partitions of $\Phi$. (A partition of $[d]$ is *connected in* $\Phi$ if each part induces a connected subgraph.)

(4) {GGfact4} If $\Phi$ is complete, $\mathcal{L}(\mathcal{H}^\infty) \cong \Pi_n$.
(5) {GGfact5} $p_{\mathcal{H}}(\lambda)$ and $w_{\mathcal{H}}(x,\lambda)$ equal the zero-free (or "balanced") chromatic and Whitney-number polynomials of $\Phi$, $\chi^*_\Phi(\lambda)$ and $w^*_\Phi(x,\lambda)$ [14, §III.5]. They can sometimes be computed by coloring [14, §III.4].
(6) {GGfact6} For an ideal flat $s$ corresponding to a partition $\pi$ of $[d]$, $1+\dim s = |\pi|$. The subarrangement $\mathcal{H}(s)$ corresponds to the induced subgraph $\Phi{:}\pi$ (the union of the subgraphs $\Phi{:}B$ induced by the parts $B \in \pi$).
(7) {GGfact7} The affine flats $t \in \mathcal{H}(s)$ correspond to the flats $A$ of $L_0$ that are balanced and which induce on $[d]$ the partition $\pi$. Furthermore, $\dim t = |\pi|$.

In the examples that have attracted the most interest so far, the graph is complete and has a special form. Assuming merely completeness, Theorem 3.1 takes the form

$$\text{(6.1)} \qquad \{\text{GGkenum}\} \sum\sum_{-1\le l<k\le d} f_{kl} x^{d-k} z^l = z^{-1}(-1)^d \sum_{\pi\in\Pi_d} w^*_{\Phi:\pi}(-x,1)(|\pi|)!(-z)^{|\pi|}.$$

Furthermore, $w^*_{\Phi:\pi} = \prod_{B\in\pi} w^*_{\Phi:B}$.

Now we apply these remarks to examples with a kind of uniformity: the gain graph belongs to a sequence $\Phi_n$ with the property that $\Phi_n{:}B \cong \Phi_{|B|}$ for each subset $B$ of the vertex set. Let $w^*_n = w^*_{\Phi_n}$. The expression in Equation 6.1 should be revised to a partially exponential generating function:

$$\text{(6.2)} \qquad \{\text{GGuenum}\} \sum\sum_{-1\le l<k\le d} f_{kl} x^{d-k} \frac{z^l}{(l+1)!} = z^{-1}(-1)^d \sum_{\pi\in\Pi_d} (-z)^{|\pi|} \prod_{B\in\pi} w^*_{|B|}(-x,1).$$

This is suitable to an overall exponential generating function, which simplifies via the exponential formula:

$$\text{(6.3)} \qquad \{\text{GGuxgf}\} \sum_{d=0}^\infty \frac{t^d}{d!} \sum\sum_{-1\le l<k\le d} f_{kl} x^{d-k} \frac{z^l}{(l+1)!} = z^{-1} \exp\left\{ -z \sum_{j=1}^\infty w^*_j(-x,1) \frac{(-t)^j}{j!} \right\}.$$

It all hinges, of course, on being able to compute $w_j$. If that is too hard, the specialization to $x=0$, which counts only regions by ideal dimension, may be tractable since $w_j(0,1) =$ the zero-free (or balanced) chromatic polynomial $\chi^*_j(\lambda) = \chi^*_{\Phi_j}(\lambda)$ (which equals the characteristic polynomial of the affine representing arrangement $\mathcal{H}$).

Our particular examples are three. The notation employed is:

* $[m,n] = \{m, m+1, \dots, n\}$.
* For $A \subseteq \mathbb{R}$, $A\vec{K}_d$ is the gain graph with edge set
$$\{(ij,\alpha) : 1 \le i < j \le d;\ \alpha \in A\},$$
where the gain of $(ij,\alpha)$ in the direction from $i$ to $j$ is $\alpha$ and $(ij,\alpha) = (ji,-\alpha)$.

Our answers are in the form of generating functions. Ehrenborg [4] has a much more explicit answer for the extended Shi arrangement; for the other examples I know of no other published solutions.

*Example* 6.1 (*The weakly-composed-partition arrangement*). This is

$$\mathcal{C}_d(m) = \{x_j - x_i = l : 1 \le i < j \le d \text{ and } -m \le l \le m\}.$$

Its gain graph is $\Phi_d = [-m,m]\vec{K}_d$. The affine flats correspond to the $m$-composed partitions of $[d]$. (See [15, Example 10.5] for more details. The first publication of the semilattices of the $\mathcal{C}_d(m)$ was by Gill [5].)

From [15, Example 10.5] we know that $\chi_j^*(\lambda) = (\lambda - jm - 1)_{j-1}$, the falling factorial, for $j > 0$. Therefore for regions the generating function (6.3) becomes

$$\sum_{d=0}^{\infty} \frac{t^d}{d!} \sum_{l=-1}^{d-1} f_{dl} \frac{z^l}{l!} = z^{-1} \exp\left\{ z \sum_{j=1}^{\infty} \binom{jm}{j-1} \frac{t^j}{j} \right\}.$$

*Example* 6.2 (*The extended Shi arrangement*). This is

$$\mathcal{S}_d(m) = \{x_j - x_i = l : 1 \leq i < j \leq d \text{ and } -m < l \leq m\}.$$

Its gain graph is $\Phi_d = [-m+1, m]\vec{K}_d$.

Headley [6] tells us that the Shi arrangement, $\mathcal{S}_d(1)$, has characteristic polynomial $\lambda(\lambda - d)^{d-1}$, which is therefore the balanced chromatic polynomial $\chi_d^*(\lambda)$. Ehrenborg, adapting the method of Athanasiadis [1, Theorem 6.5], has found the characteristic polynomial of the $m$-fold extended Shi arrangement $\mathcal{S}_d(m)$ to be $\chi_d^*(\lambda) = \lambda(\lambda - dm)^{d-1}$ [4, Corollary 2.5]. The Whitney-number polynomial of $\mathcal{S}_d(m)$ and $\Phi_d$ is computed by Athanasiadis [1, Theorem 6.5] for $\mathcal{S}_d(1)$, and for $\mathcal{S}_d(m)$ is implicit in Ehrenborg's work [4, §2]. Ehrenborg's $P(q)$, which is a function of his $n$, $r$, and rank $k$ (our $d$, $m$, and $d-k$) as well as $q$ (our $\lambda$), equals the coefficient of $x^{d-k}$ in $w_d^*(x, \lambda)$. The reason is that, when $q$ is a prime number, $P(q)$ counts points $p \in \mathbb{A}^d(\mathbb{Z}_q)$ for which the set of hyperplnaes of $\mathcal{S}_d(m)$ containing $p$ has rank $k$ (in his notation). (This observation constitutes a proof because of either [14, Corollary III.4.5] or its partial generalization in [1, Theorem 6.3].) Therefore Ehrenborg has actually proved in [4, §2] that

$$w_d^*(x, \lambda) = \lambda\big[I - x\Delta_m\big]^d (\lambda - md)^{d-1},$$

where $\Delta_m$ is the $m$-step backward difference operator, $\Delta(\lambda) = p(\lambda) - p(\lambda - m)$. Eliminating the difference operator,

$$\begin{aligned} w_d^*(x, \lambda) &= \lambda \sum_{r=0}^{d} x^r \tbinom{d}{r} \sum_{i=0}^{r} (-1))^{r-i} \binom{r}{i} \big(\lambda - (d-i)m\big)^{d-1} \\ &= \lambda \sum_{i=0}^{d} x^i (1-x)^{d-i} \binom{d}{i} \big(\lambda - (d-i)m\big)^{d-1}, \end{aligned}$$

as Athanasiadis previously found for $m = 1$ [1, Theorem 6.5].

The conclusion from all this is that our formula (6.3) gives the partially exponential generating function of all faces by dimension and ideal dimension in all ambient dimensions:

$$\begin{aligned} &\sum_{d=0}^{\infty} \frac{t^d}{d!} \sum\sum_{-1 \leq l < k \leq d} f_{kl} x^{d-k} \frac{z^l}{(l+1)!} \\ &\qquad = z^{-1} \exp\left\{ z \sum_{j=1}^{\infty} \sum_{i=0}^{j} x^i (1-x)^{j-i} \binom{j}{i} \big(m(j-i) - 1\big)^{j-1} \frac{t^j}{j!} \right\}. \end{aligned}$$

Thus we have a means of evaluating $f_{kl}(\mathcal{S}_d(m))$ that is different from Ehrenborg's in [4, §§3–4] but is based on his implicit calculation of the Whitney-number polynomial in [4, §2].

*Example* 6.3 (*The extended and 0-extended Linial arrangements*). The former is

$$\mathcal{L}_d(m) = \{x_j - x_i = l : 1 \leq i < j \leq d \text{ and } 0 < l \leq m\};$$

the latter is

$$\mathfrak{L}^0_d(m) = \mathfrak{L}_d(m) \cup \{x_j - x_i = 0 : 1 \le i < j \le d\};$$

They represent the gain graphs $\Phi_d = [1,m]\vec{K}_d$ and $\Phi^0_d = [0,m]\vec{K}_d$. Postnikov [9] found the characteristic polynomial of $\mathfrak{L}_d(1)$, the original Linial arrangement. Athanasiadis [1, Theorems 4.3 and 4.4] generalized to all; from his work it follows that

$$\chi^{0*}_d(\lambda) = \frac{\lambda}{m^d} \sum_{\substack{t_0,t_1,\dots,t_{m-1} \ge 0 \\ t_0+\cdots+t_{m-1}=d}} \binom{d}{t_0\ t_1\ \cdots\ t_{m-1}} \Big(\lambda - d - \sum_{i=1}^{m-1} it_i\Big)^{d-1}$$

and

$$\chi^{*}_d(\lambda) = \frac{\lambda+d+1}{m^{d+1}} \sum_{\substack{t_0,t_1,\dots,t_{m-1} \ge 0 \\ t_0+\cdots+t_{m-1}=d+1}} \binom{d+1}{t_0\ t_1\ \cdots\ t_{m-1}} \Big(\lambda + 1 - \sum_{i=1}^{m-1} it_i\Big)^{d}.$$

Therefore for regions of $\mathfrak{L}^0_d(m)$ the generating function of face numbers is

$$\sum_{d=0}^{\infty} \frac{t^d}{d!} \sum_{l=-1}^{d-1} f_{dl} \frac{z^l}{l!} = z^{-1} \exp\left\{ z \sum_{j=1}^{\infty} \frac{1}{m^j} \sum_{t_0+\cdots+t_{m-1}=j} \binom{j}{t_0\ t_1\ \cdots\ t_{m-1}} \Big(j - 1 + \sum_{i=1}^{m-1} it_i\Big)^{j-1} \frac{t^j}{j!} \right\}$$

and for regions of $\mathfrak{L}_d(m)$ it is

$$\sum_{d=0}^{\infty} \frac{t^d}{d!} \sum_{l=-1}^{d-1} f_{dl} \frac{z^l}{l!} = z^{-1} \exp\left\{ z \sum_{j=1}^{\infty} \frac{j+2}{m^{j+1}} \sum_{t_0+\cdots+t_{m-1}=j+1} \binom{j+1}{t_0\ t_1\ \cdots\ t_{m-1}} \Big( -2 + \sum_{i=1}^{m-1} it_i\Big)^{j} \frac{t^j}{j!} \right\}.$$

## 7. Level vs. ideal dimension

{L}

Ehrenborg defines the *level* of a set $S \subseteq \mathbb{E}^d$ to be the smallest dimension $l$ such that for some $l$-flat $t$, the distance $d(X,l)$ is bounded. That is, there is a tube

$$t(\varepsilon) = \{x \in \mathbb{E}^d : d(x,t) \le \varepsilon\}$$

around $t$ such that $X \subseteq t(\varepsilon)$. We wish to compare $l$ to the ideal dimension

$$\dim_\infty X = \dim\,[\mathrm{cl}_{\mathbb{P}}(X)]_\infty.$$

We find that they are equal for polyhedra but not for all sets.

**Lemma 7.1.** *{Lseq} Let $x, y \in h_\infty$. Suppose $\{x_i\}$ and $\{y_i\}$ are affine sequences with limits in $\mathbb{P}^d$ equal to $x$ and $y$, respectively, and furthermore suppose that $|x_i - y_i|$ is bounded. Then $x = y$.*

*Proof.* We treat $\mathbb{P}^d$ as the lower half $H$ of the unit sphere in $\mathbb{R}^{d+1}$, including the equator, modulo the antipodal map of the equator. If we project $H$ stereographically from the origin onto the plane $z = -1$ tangent to $H$ at its lowest point, then $H^\circ$ ($H$ less the equator) projects to $z = -1$, which we identify with $\mathbb{E}^d$, and the equator projects to $h_\infty$; so $H$ projects to $\mathbb{P}^d$. Call the stereographic projection mapping $\sigma$.

That a sequence $\{x_i\} \subseteq \mathbb{E}^d$ converges to $x \in h_\infty$ means that the sequence $p_i = \sigma^{-1}(x_i)$ has limit points contained in $\sigma^{-1}(x) = \{p, p^*\}$, where $p$ and $p^*$ are antipodal equatorial points. Say $p$ is a limit point. We may as well assume (by discarding points $x_i$ not in the subsequence) that $p_i \to p$. It follows from the bound on $|x_i - y_i|$ that $|p_i - q_i| \to 0$, so

$q_i = \sigma^{-1}(y_i) \to q \in \sigma^{-1}(y)$, one of the two potential limit points of $\{q_i\}$, and furthermore $q = p$. Therefore $x = y$. $\square$

**Lemma 7.2.** $\{Ltube\}$ *If $t$ is an affine $k$-flat and $\varepsilon \geq 0$, then $[\mathrm{cl}_{\mathbb{P}}(t(\varepsilon))]_\infty = [\mathrm{cl}_{\mathbb{P}}(t)]_\infty$ so also $\dim_\infty t(\varepsilon) = \dim_\infty t$.*

*Proof.* That $t \subseteq t(\varepsilon)$ implies $[\mathrm{cl}_{\mathbb{P}}(t)]_\infty \subseteq [\mathrm{cl}_{\mathbb{P}}(t(\varepsilon))]_\infty$. The task is to prove the reverse containment.

Let $x \in [\mathrm{cl}_{\mathbb{P}}(t(\varepsilon))]_\infty$. Thus $x = \lim x_i$ for a sequence $\{x_i\} \subseteq t(\varepsilon)$. Let $y_i$ be the orthogonal projection of $x_i$ onto $t$. Thus $|s_i - y_i| \leq \varepsilon$. It follows from Lemma 7.1 that $s = \lim x_i = \lim y_i \in \mathrm{cl}_{\mathbb{P}}(t)$. In other words, $[\mathrm{cl}_{\mathbb{P}}(t(\varepsilon))]_\infty \subseteq [\mathrm{cl}_{\mathbb{P}}(t)]_\infty$. $\square$

Now,

$$\begin{aligned} l &= \min\{\dim t : X \subseteq t(\varepsilon) \text{ for some } \varepsilon \geq 0\} \\ &= \min\{1 + \dim_\infty t : X \subseteq t(\varepsilon) \text{ for some } \varepsilon \geq 0\}. \end{aligned}$$

Since $X \subseteq t(\varepsilon) \implies \dim_\infty X \leq \dim_\infty t(\varepsilon) = \dim_\infty t$ by Lemma 7.2, we have

$$\dim_\infty X \leq \dim_\infty t.$$

By choosing $t$ to have dimension $l$ we get $\dim_\infty X \leq \dim_\infty t = l - 1$. Therefore:

**Lemma 7.3.** $\{Lgen\}$ *Any $X \subseteq \mathbb{E}^d$ has level $l \geq 1 + \dim_\infty X$.* $\square$

Inequality is possible if $X$ is not a polyhedron.

*Example* 7.1. {Lctrex} In $\mathbb{E}^2$ let $X$ be the convex region $|y| \leq \log x$. The only rays in $X$ are parallel to the $x$-axis. Thus $[X]_\infty =$ the ideal point on the $x$-axis; so $\dim_\infty X = 0$. On the other hand, no tube around a line contains $X$, so the level is 2.

If $X$ is a polyhedron, though, we do have equality.

**Proposition 7.4.** $\{Lpoly\}$ *If $X$ is a closed or relatively open polyhedron in $\mathbb{E}^d$, then its level $l = 1 + \dim_\infty X$.*

*Proof.* The essential fact is that a closed polyhedron in $\mathbb{E}^d$ is the intersection of finitely many closed half spaces. Thus the set $V$ of vertices of $X$ is finite. Choose the origin of coordinates to lie in the relative interior of $X$. Finally, set $\varepsilon = \max_{v \in V} |v|$.

Let $R_0$ be the union of the origin and translates of the extreme rays of $X$, each ray translated so its vertex is the origin; and let $R = \mathrm{conv}\, R_0$. Let

$$d(F, R) = \max_{x \in F} d(x, R),$$

where $d$ means distance.

**Lemma 7.5.** $\{Ldist\}$ $d(F, R) \leq \varepsilon$.

*Proof.* For each vertex, $d(v, R) \leq d(v, 0) \leq \varepsilon$. For an extreme ray $r$ with vertex $v$, if $s \in r$ then $d(x, R) \leq d(x, x - v) = d(v, 0) \leq \varepsilon$. For $x \in vw$, a bounded edge of $X$, $d(x, R) \leq \max\{d(v, R),\ d(w, R)\} \leq \varepsilon$. Since the entire 1-skeleton of $X$ lies within distance $\varepsilon$ from $R$, so does all of $X$. $\square$

**Lemma 7.6.** $\{LXinf\}$ $[X]_\infty = [R]_\infty$.

*Proof.* We have $R \subseteq X \subseteq R(\varepsilon)$, where $R(\varepsilon) = \{x \in \mathbb{E}^d : d(x, R) \leq \varepsilon\}$. Therefore $[R]_\infty \subseteq [X]_\infty \subseteq [R(\varepsilon)]_\infty$. Because $R(\varepsilon)$ contains only rays parallel to rays in $R$, $[R]_\infty = [R(\varepsilon)]_\infty$. Thus $[X]_\infty = [R]_\infty$. $\square$

Let $t = \mathrm{aff}(R)$. It is clear that $\dim_\infty t = \dim_\infty R$. Since $X \subseteq t(\varepsilon)$, the level $l$ of $X$ satisfies $l \leq \dim t = \dim_\infty t + 1 = \dim_\infty R + 1 = \dim_\infty X + 1$ (by Lemma 7.6). Combining this with Lemma 7.3, we have Proposition 7.4. $\square$

## 8. The complex complement complex

{C}

In the complex version of hyperplane arrangements, the complement of the arrangement is not divided into $|p(-1)|$ disjoint cells; rather, it is connected and has a cohomology algebra $H^*$ whose Betti numbers are the coefficients of $(-1)^d p(-y)$. (This is the remarkable theorem of Orlik and Solomon [7]; see also [8].) One would like to see a refinement of their theorem related somehow to enumeration of faces by ideal rank. In view of Theorem 3.1 it appears that one should look at the two-variable refinement of the polynomial of Equation (3.1) defined by

$$\tilde{p}(\lambda, z) = \sum_{s \in \mathcal{L}(\mathcal{H}^\infty)} p_{\mathcal{H}(s)}(1) p_{\mathcal{H}^s}(\lambda) z^{\dim s}$$

and seek a cohomological interpretation of the coefficients of $(-1)^d \tilde{p}(-y, z)$. (See Equations (4.4) and (5.3) for examples of $\tilde{p}$.) The question is whether there is an interpretation of the individual coefficients of $\tilde{p}$. It might be in terms of an interaction between the cohomology of $\mathbb{C}^d \setminus \bigcup \mathcal{H}$ and that of the induced arrangement in the infinite hyperplane of $\mathbb{CP}^d$. And are there parallel interpretations in Orlik and Solomon's two abstract algebraic constructions of $H^*$, which they call the algebras $A$ and $B$? With these unanswered questions we close.

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