The Largest Demigenus of a Bipartite Signed Graph

by

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Abstract

A graph with signed edges is *orientation embedded* in a surface when it is topologically embedded so that one trip around a closed path preserves or reverses orientation according as the path's sign product is positive or negative. We find the smallest surface within which it is possible to orientation-embed the complete bipartite signed graph $\pm K_{r,s}$, which is obtained from the complete bipartite graph $K_{r,s}$ through replacing each edge by two parallel edges, one positive and the other negative. We discuss some consequences and related problems.

Keywords: Complete bipartite signed graph, orientation embedding, antipodal embedding, forbidden minors, minimal embedding surface, maximal embedding surface.

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Introduction and Main Theorems

Given: a bipartite graph Γ (multiple edges allowed) and a list $\mathcal{L}$, possibly empty, of polygons (graphs of simple closed paths in Γ). Wanted: the smallest possible surface in which $(\Gamma, \mathcal{L})$ can be *orientation embedded*: that is, in which Γ can be topologically embedded so that one trip around a polygon in the list will preserve orientation while a trip around an unlisted polygon will reverse it. By "smallest surface" I mean a closed surface S that is minimal, among all closed surfaces in which $(\Gamma, \mathcal{L})$ can be orientation embedded, in the partial ordering given by $S_1 \leq S_2$ if S_2 is obtained from S_1 by adding some number (possibly zero) of handles or crosscaps. Equivalently, defining the *demigenus* of S_1 as the quantity $d(S_1) = 2 -$ (Euler characteristic) $= 2g + h$ if S_1 is obtained from the sphere by adding g handles and h crosscaps, S has minimum demigenus among closed surfaces in which $(\Gamma, \mathcal{L})$ embeds. Let us define the *demigenus* of $(\Gamma, \mathcal{L})$ to be the demigenus of the smallest surface within which orientation embedding is possible. Our problem is to determine this quantity. We cannot hope to solve the problem exactly for an arbitrary graph, but we can say something very general:

Theorem 1. *For bipartite graphs having r left and s right vertices and for consistent lists of orientation-preserving polygons, the largest possible demigenus is precisely $(r - 1)(s - 1) + 1$.*

Theorem 1 is a consequence of the fact that, given r and s, there is essentially one largest bipartite graph with prescribed polygon orientations. To explain this statement we have to define embedding of *signed graphs*: graphs whose edges are labelled positive or negative. If we sign each edge of a graph Γ, we require that a trip around a polygon should preserve or reverse orientation depending on whether the product of its edge signs is positive or negative, respectively. (We call this the rule of *orientation embedding of signed graphs.*) It is a fact that, for every graph Γ and any consistent prescription of orientation-preserving polygons, there is an edge signature $\sigma : E(\Gamma) \to \{+, -\}$ such that the positive polygons of the signature are exactly the prescribed polygons. Conversely, given a graph Γ and edge signature σ, the list $\mathcal{L}$ of positive polygons is a consistent orientation-preserving prescription. (This is easy to prove from standard topology; for a graph-theoretic analysis see [6, Section 2].) Writing $\Sigma = (\Gamma, \sigma)$ for the signed graph and defining its demigenus $d(\Sigma)$ to be that of $(\Gamma, \mathcal{L})$, we see that what we are really investigating is the demigenus $d(\Sigma)$ of a bipartite signed graph $\Sigma = (\Gamma, \sigma)$ in which the two color classes of vertices have sizes r and s. Obviously, every such signed graph (not having multiple edges with the same sign, which we do not need because they can be deleted without affecting the demigenus) is a subgraph of the one formed by taking both a positive and a negative edge between every left vertex and every right vertex. We call this latter a *complete bipartite signed graph* and denote it by $\pm K_{r,s}$. Theorem 1 follows immediately from the fact that $\pm K_{r,s}$ is the largest bipartite signed graph with r left and s right vertices (and no multiple edges of the same sign), the observation that $\Sigma_1 \subseteq \Sigma_2$ implies $d(\Sigma_1) \leq d(\Sigma_2)$, and the evaluation of $d(\pm K_{r,s})$.

Theorem 2. *For positive integers r and s, $d(\pm K_{r,s}) = (r-1)(s-1)+1$.*

Knowing the demigenus of a signed graph tells us the minimal embedding surface: it is $U_{d(\Sigma)}$, the sphere with $d(\Sigma)$ crosscaps, if Σ contains a negative polygon, and $T_{d(\Sigma)/2}$, the sphere with $\frac{1}{2}d(\Sigma)$ handles, otherwise. (See [5, Lemma 3.3] for the reason and references.) So we can restate Theorem 2:

The minimal surface of $\pm K_{r,s}$ is U_h where $h = (r-1)(s-1)+1$.

My reason for interest in results like these is that I believe there is a theory of orientation embedding paralleling that of topological embedding, complete with rotation systems, current and voltage graphs, forbidden minors, and surface chromatic number and Heawood-type theorems. Within this theory, $\pm K_{r,s}$ occupies the same place as does $K_{r,s}$ in the ordinary theory: it is a natural example that provides an upper bound for an important class of graphs.

Antipodal Embedding

Theorem 2 is equivalent to the proposition that $K_{2r,2s}$ can be embedded in its minimal orientable surface so as to have an involution that is *color preserving* (keeps invariant each color class) and reverses orientation (hence is free of fixed points). In order to explain and state this formally we need some terminology. An *antipodal automorphism* α of a graph $\tilde{\Gamma}$ is an involutory automorphism without fixed vertices or edges. An *antipodal autohomeomorphism* of T_g is any orientation-reversing, involutory autohomeomorphism τ. The *antipodal genus* $\gamma(\tilde{\Gamma}, \alpha)$ is the smallest g for which $\tilde{\Gamma}$ has an *antipodal embedding* in T_g with respect to α, i.e., an embedding so that some antipodal autohomeomorphism τ of T_g induces on the embedded $\tilde{\Gamma}$ precisely the antipodal graph automorphism α. Theorem 2 is a corollary of:

Lemma 3. *For each r and $s > 0$ there exists an antipodal automorphism α of $K_{2r,2s}$ such that $\gamma(K_{2r,2s}, \alpha) = \gamma(K_{2r,2s})$. That is, $K_{2r,2s}$ has a minimal orientable embedding that is antipodal with respect to α.*

Proof. We show that Ringel's minimal orientable embedding of $K_{2r,2s}$ is antipodal with respect to a suitable choice of α. (This is the hard part of the proof, because α is nonobvious.) Ringel's minimal embedding (see [1, p. 143, top] or [2, pp. 181–2]) is specified by the following rotation system R on $V(K_{2r,2s}) = \{x_1, \dots, x_{2r}, y_1, \dots, y_{2s}\}$:

$$R(x_i) = \begin{cases} y_1 y_2 \dots y_{2s-1} y_{2s}, & \text{if } i \text{ is odd,} \\ y_{2s} y_{2s-1} \dots y_2 y_1, & \text{if } i \text{ is even,} \end{cases}$$

$$R(y_j) = \begin{cases} x_1 x_2 \dots x_{2r-1} x_{2r}, & \text{if } j \text{ is odd,} \\ x_{2r} x_{2r-1} \dots x_2 x_1, & \text{if } j \text{ is even.} \end{cases}$$

We take α to be the antipodal automorphism of $K_{2r,2s}$ induced by the following fixed-point-free involutions on the $2r$ left vertices and on the $2s$ right vertices:

If r and s are odd: $x_i \leftrightarrow x_{i+r}$ and $y_j \leftrightarrow y_{j+s}$;

If r is even: $x_i \leftrightarrow x_{i+r}$ and $y_j \leftrightarrow y_{2k+1-j}$ for a fixed k;

If s is even: $x_i \leftrightarrow x_{2h+1-j}$ for a fixed h and $y_j \leftrightarrow y_{j+s}$.

Subscripts are taken modulo $2r$ for x's and $2s$ for y's. In the even cases h and k are arbitrary. It is easy to verify that α reverses the orientation of a vertex, that is,

$$R(\alpha(v)) = (\alpha(R(v)))^{-1}$$

for every $v \in V(K_{2r,2s})$.

It follows that $\gamma(K_{2r,2s}, \alpha) = \gamma(K_{2r,2s})$. Thus Lemma 3 is proved. ■

Proof of Theorem 2. We show that embeddings of $\pm K_{r,s}$ in U_{g+1} correspond to embeddings of $K_{2r,2s}$ in T_g that are antipodal with respect to α. This follows immediately from two facts: that, for any $g \geq 0$, T_g doubly covers U_{g+1} with τ (defined above) as a covering transformation, and that $K_{2r,2s}$ doubly covers $\pm K_{r,s}$ with α as a covering transformation. Therefore $d(\pm K_{r,s}) = \gamma(K_{2r,2s}, \alpha) + 1$.

Theorem 2 follows from this and Lemma 3 because $\gamma(K_{2r,2s})$ is known from Ringel's work to equal $(r-1)(s-1)$. ■

Addenda

1. *Quadrilaterality.* Since a minimal embedding of $K_{2r,2s}$ has only quadrilateral faces, the same holds true for a minimal embedding of $\pm K_{r,s}$. This is also easy to deduce directly from Euler's polyhedral formula by standard calculations as in, e.g., [4, Thm. 4.2] or [2, Thm. 4.5].

2. *Forbidden minors.* We can use our knowledge of the demigenus of $\pm K_{r,s}$ to find new forbidden minors for orientation embedding. A *forbidden minor for (orientation embedding in)* U_h is a signed graph Σ which does not embed in U_h but such that deleting or contracting any edge does allow it to embed. (For the complete definitions see [5, Sect. 10] or [6].) The symmetry group of $\pm K_{r,s}$ (to be precise, its group of "switching automorphisms") is edge transitive; therefore all edges e behave similarly. Since Euler's polyhedral

formula implies that $d(\pm K_{r,s} \setminus e) = d(\pm K_{r,s})$, $\pm K_{r,s}$ itself cannot be a forbidden minor but $\pm K_{r,s} \setminus e$ might be. I have verified that:

$\pm K_{1,s} \setminus e$ is not a forbidden minor for any $s \geq 1$;

$\pm K_{2,s} \setminus e$ is a forbidden minor for U_{s-1} if $s \geq 3$ but not if $s \leq 2$;

$\pm K_{3,s} \setminus e$ is a forbidden minor for $U_{2(s-1)}$ if $s \geq 2$;

$\pm K_{2,3} \setminus e$ and $\pm K_{2,3}/e$ are forbidden minors for the Klein bottle, U_2.

(Other forbidden minors for the Klein bottle can be found in [6, Sect. 12] but the complete list is largely unknown.)

3. *Maximum demigenus.* The *maximum demigenus* of a signed graph is the largest demigenus of a surface in which it has a cellular embedding. The maximum demigenus has been characterized in a beautiful paper by Širáň and Škoviera [3]. It is very easy to apply their Theorem 1 to $\pm K_{r,s}$. In fact, for any connected graph Γ, writing $\pm\Gamma$ for the result of replacing every edge by two parallel edges of opposite sign, their quantity $\xi(\pm\Gamma, T) = 0$ for every spanning tree T (which in this type of example we can, without loss of generality, take to be all positive). Thus we have the following result, which in particular applies to $\pm K_{r,s}$, yielding a maximum demigenus of $rs + (r-1)(s-1)$.

Theorem 4. *For a simple, connected graph Γ of order n with m edges, let Σ be $\pm\Gamma$ with a negative loop added to any number k of vertices. Then Σ has a cellular embedding with just one face, and its maximum demigenus equals $2m - n + k + 1$.*

4. *Simple bipartite graphs.* Consider a signing of a *simple* bipartite graph with color class sizes r and s. What is the maximum possible demigenus? Call it $d'_{r,s}$. Obviously, $d'_{r,s}$ is attained by some signed $K_{r,s}$, but there is no obvious candidate signing. This contrasts sharply with problems like the ones addressed here and it makes the evaluation of $d'_{r,s}$, even asymptotically, appear to be very hard indeed.

I can prove, for $r, s \geq 3$, that

$$d'_{r,s} \leq (r-1)(s-1) + 1 - \left\lceil \frac{r+s}{2} \right\rceil ,$$

so at least we know that $d'_{r,s} < d(\pm K_{r,s})$. It is easy to see that $d'_{1,s} = 0$ and $d'_{2,s} = 1$; moreover, $d'_{3,3} = 2$, $d'_{3,s} = 3$ for $s = 4, 5, 6$, $d'_{3,7} = 4$, and $d'_{3,s} = 5$ for $s = 8, 9, 10$. These fit the surprising pattern $d'_{3,s} =$ the demigenus of $K_{3,s}$ with a single negative edge. That is surely not a coincidence; but I would not expect it to hold good for larger values of r.

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