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Line Graphs of Switching Classes (Modernized)

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A *signed graph* Σ is a graph with signed edges [3]. A *switching class* (of signed graphs) is an equivalence class of signed graphs under the switching relation: $\Sigma_1 \sim \Sigma_2 \iff V(\Sigma_1) = V(\Sigma_1) = V$, $E(\Sigma_1) = E(\Sigma_2)$, and $(\exists X \subseteq V)$ an edge has the same sign in Σ_1 and Σ_2 iff both ends are in X or both in $V \backslash X$ [7]. A switching class $[\Sigma]$ is characterized by the underlying graph $\Gamma = (V, E)$ and the class of positive circuits, $\mathcal{B} = \{C : C$ is a circuit of Σ whose edge signs have positive product$\}$. The triple $\langle\Sigma\rangle = (V, E, \mathcal{B})$ is a *sign-biased graph*; it is equivalent to $[\Sigma]$. Our graphs will be finite and loopless.

Let $\langle\Sigma\rangle$ be a sign-biased graph with underlying graph $\Gamma = (V, E)$ and let

$$V_L = E,$$
$$E'_L = \{\{v, e, f\} : \ e, f \in E \ \text{ and } v \text{ is a vertex of } e \text{ and } f\}.$$

Then (V_L, E'_L) is the ordinary line graph $L(\Gamma)$. A circuit in it is called "derived" (from C_0) if its vertices are the edges of a circuit C_0 of Γ, a "vertex triangle" if it is a triangle whose vertices are edges at a common vertex in Γ. Let

$$\begin{aligned}\mathcal{B}'_L = \{C : \ & C \ \text{ is a circuit of } \ (V_L, E'_L) \ \text{ and its edge set is a sum of} \\ & \text{derived circuits and vertex triangles, all but an even number} \\ & \text{being derived from circuits in } \ \mathcal{B}(\Sigma)\ \}.\end{aligned}$$

The *unreduced line graph* $L'(\langle\Sigma\rangle)$ is $(V_L, E'_L, \mathcal{B}'_L)$. It is a sign-biased graph.

We call a digon of $L(\Gamma)$ "negative" if it is not in $\mathcal{B}'_L$. From $L'(\langle\Sigma\rangle)$ remove pairs of edges forming negative digons until this is no longer possible. The result of this "reduction" is the *(reduced) line graph* $L(\langle\Sigma\rangle)$. This is also a sign-biased graph.

Example 1. Given a graph $\Gamma = (V, E)$, let $-\Gamma$ be the graph with every edge signed negative. Then $\mathcal{B} = \{$circuits of even length$\}$. This defines a sign-biased graph $\langle -\Gamma\rangle$. Then $L(\langle -\Gamma\rangle) = \langle -L(\Gamma)\rangle$. That is, if we identify ordinary graphs as all-negative signed graphs, then ordinary line graphs are an example of our definition.

Example 2. Given also an integral weight $m(v) \geq 0$ for each vertex, let $\langle -\Gamma; m \rangle$ consist of $\langle -\Gamma \rangle$ with $m(v)$ new vertices doubly adjacent to each original vertex v, each edge pair forming a negative digon. Then $L(\langle -\Gamma; m \rangle) = \langle -L(\Gamma; m) \rangle$, where $L(\Gamma; m)$ is Hoffman's generalized line graph [5]. Thus the latter are, in our system, simply line graphs. Conversely, if $L(\langle \Sigma \rangle)$ has the form $\langle -\Gamma_0 \rangle$, then Γ_0 is a generalized line graph $L(\Gamma; m)$.

Example 3. Let D be a digraph with underlying graph Γ. A circuit is "semicoherent" if an even number of its vertices are at the head of both incident arcs or at the tail of both incident arcs. Let $\langle D \rangle = (V, E, \mathcal{B})$ where $\mathcal{B} = \{\text{semicoherent circuits}\}$. Then $L(\langle D \rangle)$ is a kind of line graph of the digraph D. The Harary–Norman line digraph of D [4] is (if one ignores the edge directions) the subgraph whose edges are the triples $\{v, e, f\}$ such that exactly one of e and f has head incident with v.

The *adjacency matrix* of Σ is the matrix $A(\Sigma)$ whose (i, j) entry is the number of positive edges less the number of negative edges between v_i and v_j, if $i \neq j$. An *incidence matrix* of Σ is a $V \times E$ matrix $M(\Sigma)$ whose (v, e) entry is

- 0 if v is not incident with e,
- ± 1 if v is incident with e, with the rule that the endpoints of e have opposite signs if e is positive and identical signs if e is negative.

An adjacency matrix $A(\langle \Sigma \rangle)$ is an adjacency matrix of any Σ' in the switching class of Σ. Thus, the eigenvalues of $A(\langle \Sigma \rangle)$ are well-defined, although $A(\langle \Sigma \rangle)$ is not. An incidence matrix $M(\langle \Sigma \rangle)$ is any incidence matrix of Σ. Since $A(L(\langle \Sigma \rangle)) = 2I - M(\langle \Sigma \rangle)^T M(\langle \Sigma \rangle)$, we have

Theorem 1. *The eigenvalues of $A(L(\langle \Sigma \rangle))$ are all ≤ 2.*

Since a graph Γ corresponds to the signed graph $-\Gamma$, this theorem is the signed-graph generalization of the fact that the eigenvalues of an ordinary line graph, or of Hoffman's generalized line graph, are ≥ -2. Call a sign-biased graph $\langle \Sigma \rangle = (V, E, \mathcal{B})$ *reduced* if every digon of Γ is in $\mathcal{B}$. The usual root system arguments, from [1], yield

Theorem 2. *With finitely many exceptions, any reduced sign-biased graph with all eigenvalues ≤ 2 is a reduced line graph.*

For sign-biased graphs there is an extension (and explanation) of Whitney's theorem on line isomorphisms. A line isomorphism is an isomorphism of line graphs. By $\pm K_4$ we mean the signed graph of order 4 with all twelve positive and negative edges.

Theorem 3. *A line isomorphism between sign-biased graphs is induced by an isomorphism, unless the graphs are subgraphs of $\langle \pm K_4 \rangle$.*

The line automorphisms of $\langle \pm K_4 \rangle$ form a degree 2 extension of the automorphism group; the extra automorphisms are due to the "triality" of the root system D_4.

Theorem 4. *A reduced sign-biased graph is a reduced line graph if, and only if, all its induced subgraphs of orders up to 6 are.*

This characterization implies the existence of a similar one for ordinary graphs that are generalized line graphs; but it does not provide the list of forbidden induced subgraphs. That list for line graphs of switching classes is in [2] (where it is expressed in terms of signed graphs); the list for generalized line graphs is in [6].

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