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The Graph Theory of Local Multiary Quasigroups

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Ideas, definitions, problems, NO theorems yet!

Fundamental Problem:
Generalize the structure theory of multiary quasigroups to continuous and differentiable *local multiary quasigroups*.

Reducible means the n-ary quasigroup $x_0 = q(x_1, x_2, \ldots, x_n)$ has an expression in terms of multiary quasigroups of lower arities.

Known Structure Theory:

(1) q is completely reducible $\iff$ it is essentially an iterated group operation. (Aczél, Belousov, and Hosszú)

(2) q is completely reducible $\iff$ its factorization graph is 3-connected. (Zaslavsky)

(3) q is completely reducible $\iff$ all its ternary retracts are essentially iterated groups. (Zaslavsky)

(4) If q is reducible, it has a unique maximal reduction. (Belousov)

(5) Algebra of quasigroups, used to prove (4). (Belousov)

(6) Graph-theoretical description of quasigroups, used to prove (2, 3, 4) and a precise decomposition theory. (Zaslavsky)

New Idea:
Treat local quasigroups by adding topology to the graph theory.

Outline of my talk:

(1) Background, more background, and still more background:
- Quasigroups.
- Continuous and local quasigroups.
- Graph theory (biased graphs).
- Representation of quasigroups by means of biased graphs.

(2) New definitions:
- Topological biased graphs (representing continuous quasigroups).
- Local biased graphs (representing local quasigroups).

(3) Speculation:
- The reducibility theorems for multiary quasigroups generalize to multiary local quasigroups with similar proofs.
- Possibly, generalize local quasigroups using graph theory.

(No theorems yet. Very sorry. Is anyone interested?)

Definition: n-**ary** (or **multiary**) **quasigroup** $\mathfrak{Q}$
(for short: **quasigroup**):

- a set $\mathfrak{Q}$,
- a function $q : \mathfrak{Q}^n \to \mathfrak{Q}$, such that
- there exist functions $q_i : \mathfrak{Q}^n \to \mathfrak{Q}$ for $i = 1, \dots, n$ such that

$$q(x_1, x_2, \dots, x_n) = x_0 \iff q_i(x_0, x_1, \dots, \hat{x}_i, \dots, x_n) = x_i.$$

($\hat{x}_i$ means: omit x_i.)

Write $q_0 := q$ for symmetry.

Example 1: **Iterated group**:
$\mathfrak{Q}$ = group with operation $*$,
$x_0 = q(x_1, \dots, x_n) := x_1 * x_2 * \cdots * x_n$.

Definition: **Isotopy**:
q and q' are *isotopic* if we can get q from q' by changing the names (separately) on each copy of $\mathfrak{Q}$ in $q' : \mathfrak{Q}^n \to \mathfrak{Q}$.

Example 2: **Iterated group isotope**:
Any q that is isotopic to an iterated group.

Definition: **Continuous quasigroup**:
$\exists$ topology on $\mathfrak{Q}$ such that

- all q_i are continuous.

Definition: **Differentiable quasigroup**:
$\exists$ differential structure on $\mathfrak{Q}$ such that

- all q_i are differentiable.

Definition: **Local quasigroup**:
Like a continuous quasigroup, but local:

- each q_i is defined on an *open subset* $D_i \subseteq \mathfrak{Q}^n$, and
- each q_i is continuous.

Example 3:
$\mathfrak{Q} = \mathbb{R}, \qquad x_0 = q(x_1, \ldots, x_n) = x_1 + x_2 + \cdots + x_n + 5.$

Example 4 (more general):
$\mathfrak{Q} = \mathbb{R}, \quad F(x_0, x_1, \ldots, x_n) =$ a continuous function such that

$$F(x_0, x_1, \ldots, x_n) = 0$$

determines the value of each x_i (continuously) when all other x_j are given; then $q_i(x_0, x_1, \ldots, \hat{x}_i, \ldots, x_n) =$ that value of x_i.

Definition: **Factorization**:

$$(1)\qquad \begin{aligned} &q(x_1, \ldots, x_n) \\ &= q'\big(x_1, \ldots, x_i,\ q''(x_{i+1}, \ldots, x_j),\ x_{j+1}, \ldots, x_n\big), \end{aligned}$$

where $1 \leq i < j-1 \leq n-1$.

Fact: Iterated groups and their isotopes are the only quasigroups (with $n \geq 3$) that factor completely. (Easy consequence of Aczél, Belousov, and Hosszú 1960.)

Definition: **Factorization graph**:

- Vertex set: $\{v_0, v_1, \ldots, v_n\}$.
- Edge set with two parts:
  - $*$ a circle of edges $v_0v_1,\ v_1, v_2,\ \ldots,\ v_{n-1}v_n,\ v_nv_0$, with $v_0v_1 \leftrightarrow x_1,\ \ldots,\ v_{n-1}v_n \leftrightarrow x_n,$ and $v_nv_0 \leftrightarrow x_0$;
  - $*$ factorization edges v_iv_j whenever there exists a factorization (1).

Fact: Iterated groups and their isotopes are the only quasigroups (with $n \geq 3$) whose factorization graphs are complete (have all edges).

Definition: **Biased graph**:

- a graph (V, E) with vertex set V and edge set E,
- a class $\mathcal{B}$ of distinguished circles (called *balanced circles*) such that
  in each subgraph composed of 3 internally disjoint paths from vertex v to vertex w, the number of circles that belong to $\mathcal{B}$ is either 0, 1, or 3 (but not 2).

Definition (basic object): **Balanced edge set**:
a subset $S \subseteq E$ such that every circle in S belongs to the distinguished class $\mathcal{B}$.

Definition: $\mathcal{E}^{\mathrm{b}}$ = the class of balanced edge sets.

A biased graph is a graph with additional structure. What is it good for?

Combinatorial geometry.
Optimization.
Various fun problems in graph theory.
$\longrightarrow$ Multiary quasigroups.

Representation of a quasigroup by a biased graph:

Given: an n-ary quasigroup $\mathfrak{Q}$ (with operation q).

Construct: a biased graph $\Omega(\mathfrak{Q})$:

(1) Start with the graph C_{n+1}, which is the circle
$$v_0v_1,\ v_1, v_2,\ \ldots,\ v_{n-1}v_n,\ v_nv_0$$
on the vertex set $\{v_0, v_1, \ldots, v_n\}$.

(2) "Thicken" each edge $v_{i-1}v_i$ in C_{n+1} by replacing it by the set of parallel edges $\{v_{i-1}v_i\} \times \mathfrak{Q}$. Each edge represents an element of $\mathfrak{Q}$ which is a possible value of x_i.

(3) This gives a circle with parallel edges. There are $|\mathfrak{Q}|^{n+1}$ different circles of length $n+1$ contained in this graph.

(4) Each circle C of length $n+1$ corresponds to a choice of values for x_0 (depending on which edge v_nv_0 is in C), x_1 (depending on which edge v_0v_1 is in C), etc. Let

$\mathcal{B} := \{C : \text{the corresponding values satisfy } x_0 = q(x_1, \ldots, x_n)\}$.

(5) Forget the quasigroup labels on the edges.

From this biased graph $\Omega(\mathfrak{Q})$, the operation q can be reconstructed uniquely up to isotopy. (Isotopic operations cannot be distinguished, because the labelling of edges was forgotten.)

Definition: Let Δ be a graph with no multiple edges. A biased graph Ω is a **biased expansion** of Δ if:

- Ω is a biased graph.
- There is a *projection mapping* $p : \Omega \to \Delta$ which is the identity on vertices and is surjective on edges.
  (Therefore, $V(\Omega) = V(\Delta)$, and the same pairs of vertices are adjacent.)
- For every circle C in Δ and every edge $vw \in C$, and for every way to choose a path $\tilde{P}$ in Ω that projects one-to-one onto the path $C \setminus vw \subseteq \Delta$, there is exactly one choice of edge $\tilde{e} \in p^{-1}(vw)$ such that $\tilde{P} \cup \tilde{e}$ is balanced.

Definition: A **group expansion** of Δ is a biased graph $\langle \mathfrak{G}\Delta \rangle$ that is constructed with a group $\mathfrak{G}$:

- The vertex set is $V(\Delta)$.
- Orient the edges of Δ arbitrarily.
- The edge set of $\langle \mathfrak{G}\Delta \rangle$ is $\mathfrak{G} \times E(\Delta)$, with the same orientations as in Δ.
- A circle $(g_1, e_1)(g_2, e_2) \cdots (g_l, e_l)$ is balanced if and only if $g_1 g_2 \cdots g_l = 1$ in $\mathfrak{G}$. (If you go against the orientation of the edge, use g_i^{-1} instead of g_i.)

A group expansion is a particular kind of biased expansion.

Properties of biased expansions and quasigroups:

(X1) The biased expansions of C_{n+1} are in one-to-one correspondence to the isotopy classes of n-ary quasigroups.
- $\Omega(\mathfrak{Q})$ is a biased expansion of C_{n+1}.
- Every biased expansion of C_{n+1} is $\Omega(\mathfrak{Q})$ for some n-ary quasigroup $\mathfrak{Q}$.

(X2) $\Omega(\mathfrak{Q})$ is a group expansion with group $\mathfrak{G} \iff$
$\mathfrak{Q}$ is an iterated group isotope with the group $\mathfrak{G} \iff$
the factorization graph is complete.

(X3) $\Omega(\mathfrak{Q})$ extends to a biased expansion $\hat{\Omega}$ of the factorization graph $\Delta(\mathfrak{Q})$, with the same circle edges and with new (multiple) edges for the factorization edges in $\Delta(\mathfrak{Q})$.

(X4) Each parallel class of new edges, corresponding to a factorization edge v_iv_j, determines the factor quasigroups in (1).

What this means:

(A) *There is no difference, except notation, between biased expansions of a circle graph, and isotopy classes of quasigroups.*

(B) *The factorization graph of a quasigroup corresponds to extending the biased expansion to the most possible vertex pairs without adding vertices.*

Example 3 again:

$$\mathfrak{Q} = \mathbb{R}, \qquad x_0 = q(x_1, \ldots, x_n) = x_1 + x_2 + \cdots + x_n + 5.$$

The biased graph Ω_3 that corresponds to this quasigroup has

- vertex set $V = \{0, 1, \ldots, n\}$,
- edge set
  $$E = \{(r, v_{i-1}, v_i) : r \in \mathbb{R},\ i = 0, 1, \ldots, n \ (\mathrm{mod}\ n+1)\},$$
- balanced circle class
  $$\mathcal{B} = \{(r_1, v_0, v_1) \cdots (r_n, v_{n-1}, v_n)(r_0, v_n, v_0) : \\ -r_0 + r_1 + \cdots + r_n + 5 = 0\}.$$

Ω_3 is a group expansion because one can rename the r_0 values to be $r_0' = 5 - r_0$ (isotopy), then take the group to be $(\mathbb{R}, +)$. That gives the same edges and the same balanced circles.

We expect Ω_3 to be a group expansion, because it extends to a biased expansion $\hat{\Omega}_3$ of the factorization graph $\Delta(\mathfrak{Q})$ by (X3), and $\Delta(\mathfrak{Q})$ is complete. Then $\hat{\Omega}_3$ is a group expansion by (X2).

$\Delta(\mathfrak{Q})$ is complete because every possible factorization exists:

$$q(x_1, \ldots, x_n) = x_1 + \cdots + x_i \\ + q''(x_{i+1}, \ldots, x_j) + x_{j+1} + \cdots + x_n + 5$$

where $q''(x_{i+1}, \ldots, x_j) := x_{i+1} + \cdots + x_j$.

Definition:
A graph is **2-connected** if it is connected, it has at least 3 vertices, and deleting any one vertex does not disconnect it.

It is **3-connected** if it is connected, it is has at least 4 vertices, and, if any 2 vertices are deleted (with their edges), the graph remains connected.

Note that a graph with 3 vertices is not 3-connected. Therefore, the next theorem will not apply to it.

Main Theorem of Biased Expansions. *If Ω is a biased expansion of a 3-connected graph Δ, then Ω is a group expansion.*

Idea of Proof: The technique used is to treat as equivalent two paths, P and P', with the same endpoints, such that their symmetric difference (as edge sets) is a balanced circle. In some undefined weak sense, they are "homotopic".

We consider an arbitrary biased expansion Ω of a 2-connected graph Δ. Here are two of the main lemmas:

Lemma 1 (Unique Maximal Extension). Let $\Delta', \Delta'' \supseteq \Delta$ have the same vertex set as Δ. Assume Ω extends to a biased expansion Ω' of Δ'. Then:

(a) Ω' is uniquely determined by Ω and Δ'.
(b) If Ω also extends to a biased expansion Ω'' of Δ'', then it extends to a biased expansion of $\Delta' \cup \Delta''$.

Lemma 2 (Theta Extension). If Δ consists of 3 internally disjoint paths joining two vertices v and w, then Ω extends to a biased expansion of $\Delta \cup vw$.

Main Theorem of Quasigroups. *If the factorization graph of a multiary quasigroup $\mathfrak{Q}$ is 3-connected, then $\mathfrak{Q}$ is an iterated group isotope.*

Proof. $\Omega(\mathfrak{Q})$, the biased expansion of C_{n+1} determined by $\mathfrak{Q}$, extends to a biased expansion of the 3-connected factorization graph $\Delta(\mathfrak{Q})$. By the Main Theorem of Biased Graphs, $\Omega(\mathfrak{Q})$ is a group expansion. By the correspondence with quasigroups, $\mathfrak{Q}$ is an iterated group isotope. □

Corollary (Dörnte 1928). An n-ary group (i.e., associative n-ary quasigroup) is an iterated group isotope.

Proof. The definition of "associativity" (which I omit) makes the factorization graph obviously 3-connected. □

There are other corollaries of similar nature, i.e., immediate deductions of some known factorization theorems. However, my method becomes difficult when one wants results that are about actual quasigroups, not isotopy classes.

I state two further results. Their proofs are similar to the preceding ones. I regretfully omit one definition from graph theory.

Second Theorem of Biased Expansions. *If Ω is a biased expansion of a graph Δ with at least 4 vertices, and if every 4-vertex contraction is a group expansion, then Ω is a group expansion.*

This immediately implies the fundamental nature of ternary retracts of an n-ary quasigroup.

Definition: A **ternary retract** of an n-ary quasigroup (with $n \geq 3$) is obtained by holding constant all but 3 of the n independent variables.

Second Theorem of Quasigroups. *If $\mathfrak{Q}$ is an n-ary quasigroup with $n \geq 4$, and if every ternary retract of $\mathfrak{Q}$ is an iterated group isotope, then $\mathfrak{Q}$ is an iterated group isotope.*

Main Problem:
Does any of this apply to local quasigroups?

Idea and Goals:

- Define continuous and local biased graphs and biased expansions.
- Prove local analogs of the graph-theory lemmas and theorems.
- Use them to deduce local versions of the preceding quasigroup theorems.
- Generalize local quasigroups using graph theory.

This is a plan. Right now, I have only some definitions.

Given: A biased graph Ω.

Definition: $\mathcal{E}^{\mathrm{b}}$ is the class of balanced edge sets.

Definition: **Pencil**:
a nonempty set
$$E_{vw} = \{\text{edges with endpoints } v, w\}.$$
Every pair of adjacent vertices has a pencil.

Definition: **Balance-closure**:
The operation on balanced edge sets, $\mathrm{bcl} : \mathcal{E}^{\mathrm{b}} \to \mathcal{E}^{\mathrm{b}}$, defined by
$$\mathrm{bcl}(S) := S \cup \{e : \exists \text{ balanced circle } C \text{ with } e \in C \subseteq S \cup e\}.$$
This is an abstract closure, not a topological closure.

Properties:

- bcl, together with a balanced, connected edge set S that contains both an edge e_{uv} and an edge e_{wx}, induces a *partial injection* between pairs of pencils,
$$\alpha_{S,uv,wx} : E_{uv} \to E_{wx}.$$
(These are partial functions, not functions.)
- $\alpha_{S,uv,wx}$ and $\alpha_{S,wx,uv}$ are inverse functions where the composition is defined.
- If Ω is a biased expansion, each partial function $\alpha_{S,uv,wx} : E_{uv} \to E_{wx}$ is a bijection.

Definition: **Topological biased graph**:

- Biased graph Ω.
- Choose a topology $\mathcal{T}_{vw}$ on each pencil of parallel edges, E_{vw}.
- Subproduct topology on edge sets and therefore on the class of balanced edge sets, $\mathcal{E}^{\mathrm{b}}$. (Similar to a product topology.)
- bcl must be continuous.

Properties:

- The partial functions $E_{uv} \to E_{wx}$ are continuous.
- If Ω is a biased expansion, the partial functions $E_{uv} \to E_{wx}$ are homeomorphisms.
  Thus, all pencils have the same topological type.

Example 3 yet again:
$\mathfrak{Q} = \mathbb{R}, \qquad x_0 = q(x_1, \ldots, x_n) = x_1 + x_2 + \cdots + x_n + 5.$

$\mathbb{R}$ has the usual topology. Thus, $\times_{i=0}^{n} E_{v_{i-1}v_i}$ gets a product topology $\cong \mathbb{R}^{n+1}$. The subset $\mathcal{E}^{\mathrm{b}}$ inherits this topology. In fact, one can show that $\mathcal{E}^{\mathrm{b}} \cong \mathbb{R}^n$ as a subspace of $\mathbb{R}^{n+1}$.

Definition: **Local quasigroup**:

- a topological space $\mathfrak{Q}$,
- open subsets $D_i \subseteq \mathfrak{Q}^n$ for $i = 0, 1, \ldots, n$,
- continuous functions $q_i : D_i \to \mathfrak{Q}$ such that

$$q_j(x_0, \ldots, \hat{x}_j, \ldots, x_n) = x_j \\ \iff q_i(x_0, \ldots, \hat{x}_i, \ldots, x_n) = x_i$$

  when $(x_0, \ldots, \hat{x}_i, \ldots, x_n) \in D_i$
  and $(x_0, \ldots, \hat{x}_j, \ldots, x_n) \in D_j$.

($\hat{x}_i$ means: omit x_i.)

Properties:

- Each q_i may have a different domain.
- It is a *partial function* on $\mathfrak{Q}^n$.
- Each q_i defines retracts

$$q_{ij}(x_j) := q_i(a_0, \ldots, x_j, \ldots, \hat{x}_i, \ldots, a_n) : D_j \to D_i,$$

  which are partial functions on $\mathfrak{Q}$ and are *local homeomorphisms.*

Example 5 (generalized Example 4):

$$F : D_0 \times D_1 \times \cdots \times D_n \to D',$$

a continuous function such that a fixed level set

$$F(x_0, x_1, \ldots, x_n) = \text{ fixed constant}$$

determines the value of each x_i (continuously) when all other x_j are given. Define

$$q_i(x_0, x_1, \ldots, \hat{x}_i, \ldots, x_n) = \text{ that value of } x_i.$$

Modelling Example 5 with biased graphs:
We follow the pattern for quasigroups. We take Ω such that

- $V = \{v_0, v_1, \ldots, v_n\}$, and
- every edge has endpoints v_{i-1}, v_i for some i (mod $n+1$), so every pencil is E_{v_{i-1},v_i}.

Then we identify $D_i = E_{v_{i-1},v_i}$ and the fixed level set of F with

$$\mathcal{E}_0^{\mathrm{b}} := \{S \subseteq E : S \in \mathcal{E}^{\mathrm{b}} \text{ and } S \text{ connects all the vertices}\}.$$

Goal:
To model (and generalize?) local quasigroups (and in particular this example) with topological biased graphs and biased expansions.

Master Definition (and end of talk):
Local biased graph:

- A topological biased graph Ω.
- Every partial function $\alpha_{S,uv,wx} : E_{uv} \to E_{wx}$ (determined by bcl as previously described) must be a *local homeomorphism.*

The last property is the special property of the "local" topology that makes a local biased graph similar to a local quasigroup.

This is the definition. *Next?*

- Prove it is internally consistent and includes all local quasigroups. (In progress.)
- Is it more general than local quasigroups?

$\to$ Does it have factorization properties of quasigroups?
$\to$ Does it have interesting properties of local quasigroups?
$\to$ Adapt to local differentiable quasigroups and local differentiable biased graphs.

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