An Enumerative Geometry for Magic and Magilatin Labellings

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Dedicated to the memory of Claudia Zaslavsky, 1917–2006.

Abstract. A *magic labelling* of a set system is a labelling of its points by distinct positive integers so that every set of the system has the same sum, the *magic sum*. Examples are magic squares (the sets are the rows, columns, and diagonals) and semimagic squares (the same, but without the diagonals). A *magilatin labelling* is like a magic labelling but the values need be distinct only within each set. We show that the number of $n \times n$ magic or magilatin labellings is a quasipolynomial function of the magic sum, and also of an upper bound on the entries in the square. Our results differ from previous ones because we require that the entries in the square all be different from each other, and because we derive our results not by *ad hoc* reasoning but from a general theory of counting lattice points in rational inside-out polytopes. We also generalize from set systems to rational linear forms.

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1. It's all kinds of magic

We offer a theory for counting magic squares and their innumerable relatives: semimagic and pandiagonal magic squares, magic cubes and hypercubes, magic graphs, and magical oddities like circles, spheres, and stars [1, 2].

A *magic square* is an $n \times n$ array of distinct positive integers whose sum along any *line* (row, column, or main diagonal) is the same number, the *magic sum*. Magic squares date back to China in the first millenium B.C.E. [7], came in the first millenium C.E. to the Islamic world and India [8], and passed to Europe in the later Middle Ages [8] and to sub-Saharan Africa not much after [18]. The contents of a magic square have varied with time and writer;

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usually they have been the first n^2 consecutive positive integers (*standard* squares), but often any arithmetic sequence and sometimes fairly arbitrary numbers. The fixed ideas are that they are integers, positive (or rarely, nonnegative), and distinct. Even the mathematical treatises [1, 2, 6] take positivity and distinctness so much for granted as never to mention them.

In the last century mathematicians took an interest in results about the number of squares with a fixed magic sum, but with simplifications. Thus diagonal sums were usually omitted and, most significantly, the fundamental requirement of distinctness was almost invariably neglected. Our work, however, follows tradition by adhering to the distinctness requirement. We shall call a square *magic* (or *strongly magic*) if its entries are distinct and *weakly magic* if they need not be distinct; and strongly or weakly *semimagic* if the diagonals are ignored.

The literature on exact formulas examines the functions $W(t)$ and $W_0(t)$ that count positive and nonnegative weak semimagic or magic squares with magic sum t. These functions are amenable to analysis in terms of Ehrhart theory [9]. (See for example [10, 14] for semimagic squares; for magic squares see [3].) One treats a square with upper bound or magic sum t as an integer vector $x \in t \cdot [0,1]^{n^2}$, confined to the subspace ts, where $t = 1, 2, 3, \ldots$ and

$$s := \{x \in \mathbb{R}^{n^2} : \text{ all line sums equal } 1\},$$

the *magic subspace*. (Exactly which subspace this is depends on whether we treat squares that are semimagic, magic, pandiagonal magic [where we include among the line sums the wrapped diagonals], or of another type.) Thus a square x is an integer lattice point in the t-dilate tP of the *magic polytope* $P := [0,1]^{n^2} \cap s$; moreover, $x \in tP^\circ$, the relative interior of P, if and only if the square is positive.

Ehrhart's fundamental results on integer-point enumeration in polytopes [9] give much insight. Ehrhart theory implies that W and W_0 are quasipolynomials in t. (A *quasipolynomial* is a function $Q(t) = \sum_0^d c_i t^i$ with coefficients c_i that are periodic functions of t, so that Q is a polynomial on each residue class modulo some integer, called the *period*; these polynomials are the *constituents* of Q.) The quasipolynomials are polynomials (that is, the periods are 1) in the semimagic case, because the matrix that defines s is totally unimodular so the vertices of P are all integral. In the magic case this is unfortunately not so and the period is not easy to calculate.

Still there were no exact (theoretical) formulas for strong squares (not even in the comprehensive tome [13]), with the exception of Stanley's [16, Exercise 4.10]. With the theory of *inside-out polytopes* [4] we can attack this and related counting problems in a systematic way, obtaining a general result about magic counting functions and an interpretation of reciprocity that leads to a new kind of question about permutations. In inside-out theory we supplement the polytope $P = [0,1]^{n^2} \cap s$ with the pair-equality hyperplane arrangement

$$\mathcal{H} := \mathcal{H}[K_{n^2}]^s = \{h_{ij} \cap s : i < j \leq n^2\},$$

where h_{ij} is the hyperplane $x_i = x_j$, $\mathcal{H}[\Gamma] := \{h_{ij} : ij \in E\}$ is the hyperplane arrangement of the graph Γ with edge set E, and K_d denotes the complete graph on d nodes. The number of $n \times n$ squares corresponding to s with magic sum t is the number of integer points in $t\,(P^\circ \setminus \bigcup \mathcal{H})$. This is a quasipolynomial in t by the general theory of inside-out polytopes. Then inside-out reciprocity [4] gives the enumeration of weak nonnegative squares with multiplicity; this is reminiscent of Stanley's theorem on acylic orientations [15].

Another famous kind of square is latin squares and their relatives. Here each line has n different numbers. In a *latin square* these n numbers are the same in every line and are normally taken to be the first n positive integers. In any latin square in this broad meaning, every line has the same sum. Suppose we add this property to the definition of a latin square but we loosen the restriction on the entries, so that the square is filled with positive integers having equal row and column sums. We call such squares *magilatin*. Then a magilatin square is a point in $\mathbb{Z}^{n^2}$; the only difference between a semimagic and a magilatin strong square is that we assume fewer nonequalities between the entries; while in a semimagic square each entry must differ from every other, in a magilatin square it must differ only from those that are collinear with it, a *line* being a row or column. As with magic and semimagic squares, inside-out polytope theory yields theorems about the number of magilatin squares as a function either of the magic sum or of the largest allowed value of an entry in the square.

There is a parallel generalization of latin rectangles. A *latin rectangle* is an $m \times n$ rectangular array filled by n symbols, none repeated in a row or column. The asymptotic numbers of latin squares and rectangles of given dimensions have been the subject of many studies. Our geometric counting method leads in a different direction that, as far as we know, has not been studied. Define a *magilatin rectangle* to be an $m \times n$ array of positive integers with all row sums equal and all column sums equal, but with no repeated values in any row or column. We discuss the number of magilatin squares or rectangles of fixed dimensions as we vary the maximum permitted value.

The magic and latin properties generalize far beyond squares and rectangles. Semimagic and pandiagonal magic squares suggest a general picture: that of a *covering clutter*, consisting of a finite set X of points together with a family $\mathcal{L}$ of subsets, called *lines* for no particular reason, of which none contains any other and none is empty, and whose union is X. We want to assign positive integers to X so that all line sums are equal to a single number. Such a labelling is called a weakly or strongly magic or latin labelling of the covering clutter, depending on the particular requirements. There are many interesting examples, that we cannot treat individually here; but we mention three:

- A finite affine or projective geometry, the "lines" being the subspaces of any fixed dimension; more generally, a block design.
- An $n \times n \times \cdots \times n$ hypercubical array.
- A *k-net*, where the lines fall into k parallel classes (with $k \geq 2$) so that each point belongs to a unique line in each parallel class. (A semimagic square is a 2-net and a pandiagonal square is a kind of 4-net.)

These examples have lines of equal size, a property that has advantages but is not necessary for the theory to apply. Consider for instance *magic graphs* [11, Section 5.1], where the edges are assigned positive integers so that the sum of labels of all edges incident to a node is the same for every node. The lines are the vertex stars; they have equal size only for a regular graph.

These ideas generalize still further. Take rational linear forms $f_1, f_2, \ldots, f_m$. A magic labelling of $[d] := \{1, 2, \ldots, d\}$ with respect to $f_1, f_2, \ldots, f_m$ is an integer point $x \in \mathbb{R}^d$ such that all the values $f_i(x)$ are equal to the same number. The analog here of a covering clutter in which all lines have the same size is a system of forms for which all values $f_i(\mathbf{1})$ (the *weights*; $\mathbf{1}$ is the vector of all ones) are equal; such systems have nice properties.

We treat two distinct interesting approaches to enumeration. Traditionally, magic and semimagic squares have been counted with the magic sum as the parameter (due to its geometrical interpretation we call this *affine* counting); but another tack is to take as parameter the maximum allowed value of a label (which we call *cubical* counting). These same two systems apply to latinity. In our treatment we develop both counting systems equally.

The reader may wonder how practical our counting method is. We believe it is relatively feasible. In [5] we apply it to solve in utter detail six problems of 3×3 squares: magic, semimagic, and magilatin (all strong), counted both cubically and affinely.

2. Inside-out polytopes take the stage

The theory of inside-out polytopes [4] is designed to count those points of the integral lattice $\mathbb{Z}^d$ that are contained in a rational convex polytope P but not in a rational affine hyperplane arrangement $\mathcal{H}$, that is, where each hyperplane is spanned by the rational points it contains. We call $(P, \mathcal{H})$ a *rational inside-out polytope*, *closed* if P is closed. We shall always assume P is closed, except when we specifically state otherwise. The affine span $\operatorname{aff} P$ may be a proper subspace of $\mathbb{R}^d$.

A *region* of $\mathcal{H}$ is a connected component of $\mathbb{R}^d \setminus \bigcup \mathcal{H}$. A *closed region* is the closure of a region. A *region* of $(P, \mathcal{H})$ is the nonempty intersection of a region of $\mathcal{H}$ with P. A *vertex* of $(P, \mathcal{H})$ is a vertex of any such region. Note that a closed region of $(P, \mathcal{H})$ is the closure of an open region of $(P, \mathcal{H})$ and therefore meets the relative interior P°. The *denominator* of $(P, \mathcal{H})$ is the smallest positive integer t for which $t^{-1}\mathbb{Z}^d$ contains every vertex of $(P, \mathcal{H})$.

The fundamental counting functions associated with $(P, \mathcal{H})$ are the *closed Ehrhart quasipolynomial*,

$$E_{P,\mathcal{H}}(t) := \sum_{x \in t^{-1}\mathbb{Z}^d} m_{P,\mathcal{H}}(x),$$

where P is closed and the *multiplicity* $m_{P,\mathcal{H}}(x)$ of $x \in \mathbb{R}^d$ with respect to $\mathcal{H}$ is defined through

$$m_{P,\mathcal{H}}(x) := \begin{cases} \text{the number of closed regions of } (P, \mathcal{H}) \text{ that contain } x, & \text{if } x \in P, \\ 0, & \text{if } x \notin P, \end{cases}$$

and the *open Ehrhart quasipolynomial*,

$$E^\circ_{P^\circ,\mathcal{H}}(t) := \# \left(t^{-1}\mathbb{Z}^d \cap [P^\circ \setminus \textstyle\bigcup \mathcal{H}] \right).$$

We denote by $\operatorname{vol} P$ the volume of P normalized with respect to $\mathbb{Z}^d \cap \operatorname{aff} P$, that is, we take the volume of a fundamental domain of $\mathbb{Z}^d \cap \operatorname{aff} P$ to be 1. When P is full dimensional this is the ordinary volume.

Theorem 2.1 ([4, Theorem 4.1])**.** *If $(P, \mathcal{H})$ is a closed, full-dimensional, rational inside-out polytope in $\mathbb{R}^d$, then $E_{P,\mathcal{H}}(t)$ and $E^\circ_{P^\circ,\mathcal{H}}(t)$ are quasipolynomials in t that satisfy the reciprocity law $E^\circ_{P^\circ,\mathcal{H}}(t) = (-1)^d E_{P,\mathcal{H}}(-t)$, with period equal to a divisor of the denominator of $(P, \mathcal{H})$, with leading term $(\operatorname{vol} P)t^d$, and with constant term $E_{P,\mathcal{H}}(0)$ equal to the number of regions of $(P, \mathcal{H})$.*

In particular, if $(P, \mathcal{H})$ is integral then $E_{P,\mathcal{H}}$ and $E^\circ_{P^\circ,\mathcal{H}}$ are polynomials.

The *Möbius function* of a finite partially ordered set S is the function $\mu : S \times S \to \mathbb{Z}$ defined recursively by

$$\mu(r,s) := \begin{cases} 0 & \text{if } r \not\leq s, \\ 1 & \text{if } r = s, \\ -\sum_{r \leq u < s} \mu(r,u) & \text{if } r < s. \end{cases}$$

Sources are, *inter alia*, [12] and [16, Section 3.7].

The *intersection semilattice* of $\mathcal{H}$ is defined as

$$\mathcal{L}(\mathcal{H}) := \left\{ \bigcap \mathcal{S} : \mathcal{S} \subseteq \mathcal{H} \text{ and } \bigcap \mathcal{S} \neq \varnothing \right\},$$

ordered by reverse inclusion (so the whole space $\mathbb{R}^d$, the intersection of no hyperplanes, is the bottom element $\hat{0}$). Its members are the *flats* of $\mathcal{H}$. The *intersection poset* of $(P^\circ, \mathcal{H})$ is defined as

$$\mathcal{L}(P^\circ, \mathcal{H}) := \left\{ P^\circ \cap \bigcap \mathcal{S} : \mathcal{S} \subseteq \mathcal{H} \right\} \setminus \left\{ \varnothing \right\},$$

ordered by reverse inclusion. The intersection poset is a ranked poset and every interval is a geometric lattice. It equals $\mathcal{L}(\mathcal{H})$ if $\bigcap \mathcal{H}$ meets P°.

The arrangement *induced* by $\mathcal{H}$ on $s \subseteq \mathbb{R}^d$ is

$$\mathcal{H}^s := \{ h \cap s : h \in \mathcal{H},\ h \not\supseteq s,\ h \cap s \neq \varnothing \}.$$

For the second theorem we need the notion of transversality. The arrangement $\mathcal{H}$ is *transverse* to P if every flat $u \in \mathcal{L}(\mathcal{H})$ that intersects P also intersects P°, and P does not lie in any of the hyperplanes of $\mathcal{H}$. Let

$$E_P(t) := \# \left(tP \cap \mathbb{Z}^d \right),$$

the standard Ehrhart counting function (without any hyperplanes present).

Theorem 2.2 ([4, Theorem 4.2])**.** *If P and $\mathcal{H}$ are as in Theorem 2.1, then*

$$E^\circ_{P^\circ,\mathcal{H}}(t) = \sum_{u \in \mathcal{L}(P^\circ,\mathcal{H})} \mu(\hat{0}, u) E_{P^\circ \cap u}(t), \tag{2.1}$$

and if $\mathcal{H}$ is transverse to P,

$$E_{P,\mathcal{H}}(t) = \sum_{u \in \mathcal{L}(P^\circ,\mathcal{H})} |\mu(\hat{0}, u)| E_{P \cap u}(t). \tag{2.2}$$

Often the polytope is not full-dimensional. Suppose that s is any affine subspace. Its *period* $p(s)$ is the smallest positive integer p for which $p^{-1}\mathbb{Z}^d$ meets s.

Corollary 2.3 ([4, Corollary 4.3])**.** *Let P be a rational convex polytope and $\mathcal{H}$ a hyperplane arrangement in $s := \operatorname{aff} P$. Then $E_{P,\mathcal{H}}(t)$ and $E^\circ_{P^\circ,\mathcal{H}}(t)$ are quasipolynomials in t that satisfy the reciprocity law $E^\circ_{P^\circ,\mathcal{H}}(t) = (-1)^{\dim s} E_{P,\mathcal{H}}(-t)$. Their period is a multiple of $p(s)$ and a divisor of the denominator of $(P,\mathcal{H})$. If $t \equiv 0 \mod p(s)$, the leading term of $E_{P,\mathcal{H}}(t)$ is $[\operatorname{vol} p(s)P] t^{\dim s}$ and its constant term is the number of regions of $(P,\mathcal{H})$; but if $t \not\equiv 0 \mod p(s)$, then $E_{P,\mathcal{H}}(t) = E^\circ_{P^\circ,\mathcal{H}}(t) = 0$.* ■

Note that $\operatorname{vol} p(s)P$ is defined with respect to the integer lattice in the dilation $p(s)s$ of the affine subspace s.

3. Magic squares, magic labellings of covering clutters, and equal line sums

3.1. **Sorts of magic.** We pointed out in the introduction that the difference between weak and strong magic (or semimagic) squares lies in the fact that for the latter we require the entries to be distinct. We will therefore spend the beginning of this section studying the general setting of integer points in polytopes with distinct entries.

We have a convex polytope $P \subseteq \mathbb{R}^d$, spanning an affine subspace s. To ensure distinctness of the coordinates of a vector we avoid the hyperplanes of $\mathcal{H} := \mathcal{H}[K_d]^s$, the complete-graph arrangement $\mathcal{H}[K_d]$ intersected with s. Transversality of $\mathcal{H}$ and P means that, first of all, s is not a subspace of any hyperplane $x_j = x_k$ and, secondly, any flat of $\mathcal{H}[K_d]$ that meets P also meets P°. In many of the interesting special cases the latter condition is automatic.

Suppose x is a point in $\mathbb{R}^d$ whose entries are all distinct. There is a unique permutation τ of $[d]$ such that $x_{\tau 1} < x_{\tau 2} < \cdots < x_{\tau d}$. We say x *realizes* τ. We call a permutation σ *realizable* in a subset $A \subseteq s$ if there is a vector $x \in A$ that realizes it. We are interested in realizability in P, but realizability in s is simpler. Fortunately, a permutation that is realizable in s is also realizable in P if P contains a positive multiple of $\mathbf{1} := (1, 1, \ldots, 1)$, since every closed region of $\mathcal{H}$ contains $\langle \mathbf{1} \rangle$. This is the case when every form has equal positive weight.

If $x \in \mathbb{R}^d$ satisfies $x_{\sigma 1} \leq x_{\sigma 2} \leq \cdots \leq x_{\sigma d}$, we say x and σ are *compatible.*

Theorem 3.1. *Suppose $P \subseteq \mathbb{R}^d$ is a closed, rational convex polytope transverse to $\mathcal{H}[K_d]$ and $s := \operatorname{aff} P$. The number $E^\circ_{P^\circ, \mathcal{H}[K_d]^s}(t)$ of integer points in tP° with distinct entries is a quasipolynomial in positive integers t with leading term $(\operatorname{vol} P) t^{\dim s}$. Furthermore, $(-1)^{\dim s} E^\circ_{P^\circ, \mathcal{H}[K_d]^s}(-t) = E_{P, \mathcal{H}[K_d]^s}(t) :=$ the number of pairs (x, σ) consisting of an integer point $x \in tP$ and a compatible P-realizable permutation σ of $[d]$. The constant term of $E_{P, \mathcal{H}[K_d]^s}(t)$ equals the number of permutations of $[d]$ that are realizable in P.*

Proof. The first statement is a direct consequence of Corollary 2.3 along with the observation that, by transversality, a region that intersects P must also intersect P°. For the second, the multiplicity $m(x)$ of x equals the number of closed regions of $\mathcal{H}[K_d]^s$ that contain x. The regions of $\mathcal{H}[K_d]^s$ correspond to certain regions of $\mathcal{H}[K_d]$, which correspond to permutations of $[d]$. Clearly, a closed region contains x if and only if its permutation is compatible with x. Thus, $m(x)$ is the number of permutations that are both realizable in P and compatible with x. Now appeal to Corollary 2.3. ∎

Problem 3.2. The period and denominator present a puzzle. The denominator of the inside-out polytope $(P, \mathcal{H}[K_d])$ is obviously a multiple of the denominator of the standard polytope P. The first question is when the hyperplane arrangement $\mathcal{H}[K_d]$ changes this latter denominator, and in what way. As for the period, if in particular P has integral vertices then E_P is a polynomial. What conditions on P ensure that $(P, \mathcal{H}[K_d])$ is also integral, so that $E_{P, \mathcal{H}[K_d]}$ is a polynomial? That it need not be is illustrated by the simple example of the line segment from $(0, 1)$ to $(1, 0)$ in $\mathbb{R}^2$ and the hyperplane $x_1 = x_2$.

Our desire to develop the ideas behind magic squares and graphs suggests two approaches to choosing P and s. The subspace, s, represents the existence of a magic sum. The polytope, P, represents the constraints on the entries in the value vector. The magic sum constraints may be pure equalities:

(i) Set all linear forms equal to each other. (Homogeneous equations.)

Or, they may be set all equal to a controlled constant:

(ii) Set all linear forms equal to t. (Affine equations.)

(Sometimes one wants additional equations; see the discussion of centrally symmetric squares in Examples 3.9 and 3.19.) Similarly, the constraints on the components of x may be two-sided bounds:

(I) All variables x_i satisfy $0 \leq x_i \leq t$. (Cubical constraints.)

Or, the constraints may be merely nonnegativity of the variables:

(II) All variables $x_i \geq 0$. (Nonnegativity.)

We think the natural combinations are (i) with (I) and (ii) with (II) and that is how we develop the theory.

3.2. **Cubical magic.** The cubical approach to magic squares counts them by the largest allowed value of the entry in a cell; if t is the parameter, the squares counted are those with entries $0 < y_{ij} < t$.

In the general situation the magic subspace s is defined by homogeneous, rational linear equations

$$f_1(x) = f_2(x) = \cdots = f_m(x) = 0. \tag{3.1}$$

These are obviously equivalent to (i), if the forms in (3.1) are the differences of the forms of (i). The magic polytope is

$$P := [0,1]^d \cap s,$$

whence the name "cubical". The hyperplane arrangement is $\mathcal{H} := \mathcal{H}[K_d]^s$. In the example of magic squares, P is the set of all square matrices with real entries in $[0,1]$ such that all row, column, and diagonal sums are equal. For magic graphs, P is the set of edge labellings by numbers in $[0,1]$ such that all node sums are equal.

One has to be sure that P spans s.

Lemma 3.3. *If P is not contained within a coordinate hyperplane, it affinely spans the magic subspace s.*

Proof. First, P contains a point of the open hypercube $(0,1)^d$. By the hypothesis P contains points x^i (not necessarily distinct) with $0 < (x^i)_i \leq 1$. The barycenter x of these points lies in $(0,1]^d$ and in P. Then $\frac{1}{2}x \in P \cap (0,1)^d$.

Therefore P does not lie in the union of the coordinate hyperplanes and the hyperplanes $x_i = 1$. Since P is bounded by those hyperplanes, $P^\circ = (0,1)^d \cap s$, which spans s because it is nonvoid. ∎

If P is contained in a coordinate hyperplane, one should reformulate the problem with fewer variables.

Now for the main theorem on cubical magic. The magic subspace $s \subseteq \mathbb{R}^d$ is given by (3.1); the polytope is $P := [0,1]^d \cap s$. For $t = 1, 2, \ldots$ let

$B^\circ(t) :=$ the number of integer points $x \in s$ with distinct entries that satisfy $0 < x_i < t$,

and let

$B(t) :=$ the number of pairs (x, σ) consisting of an integer point $x \in s$ that satisfies $0 \leq x_i \leq t$ and a compatible s-realizable permutation σ of $[d]$.

Theorem 3.4 (Magic enumeration by bounds)**.** *Suppose $P := [0,1]^d \cap s$ does not lie within a coordinate hyperplane. Then B° and B are quasipolynomials with leading term $(\operatorname{vol} P)t^{\dim s}$ and with constant term $B(0)$ equal to the number of permutations of $[d]$ that are realizable in s. Furthermore, $(-1)^{\dim s} B^\circ(-t) = B(t)$.*

Proof. Theorem 3.1 shows that $B = E_{P,\mathcal{H}}$ and $B^\circ = E^\circ_{P^\circ,\mathcal{H}}$ are reciprocal quasipolynomials. The constant term equals the number of permutations that are realizable in P. To prove these are the same permutations that are realizable in s, we show that a permutation realized in s is realized by a point in P°.

That a permutation σ is realized in s means that it is realized by a region R of $\mathcal{H}[K_d]^s$. Each region R' of $\mathcal{H}[K_d]$ meets every neighborhood of $\langle \mathbf{1} \rangle = \bigcap \mathcal{H}[K_d]$; thus, it meets $(0,1)^d$. Consequently, $R' \cap s$, if nonvoid, intersects $(0,1)^d \cap s$, which is P° by Lemma 3.3. Choosing R' so that $R = R' \cap s$, it follows that $R \cap P^\circ$ is nonempty. Any point in it realizes σ. ■

Most interesting is the case in which all the forms f_i have weight zero. It is precisely then that $\mathcal{H}$ and P are transverse, as one can see by comparing the subspace $\bigcap \mathcal{H} = \langle \mathbf{1} \rangle \cap s$ with the definition of transversality; moreover, then P spans s.

Theorem 3.5. *With s defined by forms of weight zero, and assuming a magic labelling exists, we have*

$$B^\circ(t) = \sum_{u \in \mathcal{L}(P^\circ,\mathcal{H})} \mu(\hat{0},u) E_{(0,1)^d \cap u}(t) \ ,$$

$$B(t) = \sum_{u \in \mathcal{L}(P^\circ,\mathcal{H})} |\mu(\hat{0},u)| E_{[0,1]^d \cap u}(t) \ ,$$

where μ is the Möbius function of $\mathcal{L}(P^\circ,\mathcal{H})$.

Proof. Apply Theorem 2.2 in s. A magic labelling exists if and only if s does not lie in any hyperplane $x_j = x_k$. ■

A flat $u \in \mathcal{L}(\mathcal{H})$ has the form $v \cap s$ where v is given by a series of equations of coordinates: $x_{i_1} = x_{i_2} = \cdots = x_{i_p}$, $x_{j_1} = x_{j_2} = \cdots = x_{j_q}$, etc.; that is, v corresponds to a partition π of X. We can treat these equations as eliminating the variables $x_{i_2}, \ldots, x_{i_p}, x_{j_2}, \ldots, x_{j_q}, \ldots$ in favor of $x_{i_1}, x_{j_1}, \ldots$. With this substitution, $v = \mathbb{R}^{d'}$ for some $d' < d$ and $[0,1]^d \cap v = [0,1]^{d'}$. Then $[0,1]^d \cap u$ is essentially $[0,1]^{d'} \cap s'$, where s' is s after identifying variables. Similar remarks apply to $\mathcal{L}(P^\circ,\mathcal{H})$, since its members are the nonvoid intersections with P° of flats of $\mathcal{H}$.

We describe some of the most interesting examples, concluding with the two best known.

Example 3.6 (Lines of constant length). In a covering clutter $(X,\mathcal{L})$, suppose every line has the same number of points. Then the linear equations that express the existence of a magic sum take the form

$$\sum_{j \in L_1} x_j = \sum_{j \in L_2} x_j \text{ for all } L_1, L_2 \in \mathcal{L} \ .$$

Theorems 3.4 and 3.5 both apply: $B^\circ(t)$ and $B(t)$ are reciprocal quasipolynomials in the upper bound t, they have leading term $(\operatorname{vol} P)t^{\dim s}$, $B(0)$ is the number of permutations that can be realized by magic labellings of $(X,\mathcal{L})$, and $B(t)$ and $B^\circ(t)$ reduce through Möbius inversion to ordinary Ehrhart quasipolynomials. Examples include magic, semimagic, and pandiagonal magic squares, affine and projective planes, k-nets, and magic hypercubes with or without diagonals of various kinds, as well as magic labelling of regular graphs. One has to ask about the existence of a magic labelling. There is no known general answer, but certainly there exist magic and semimagic squares of all orders $n \geq 3$ and pandiagonal magic squares of all orders $n \geq 4$, using the standard entries $\{1, 2, \ldots, n^2\}$ if $n \not\equiv 2 \mod 4$ (see [2, pp. 203–211]).

For a general covering clutter only Theorem 3.4 applies. If it has magic labellings at all, the problems of existence and characterization of realizable permutations come to the fore. An example of nonexistence is the Fano plane. Examples of existence are magic squares of size $n \geq 3$. Regarding characterization we propose a conjecture. We may assume $X = [d]$. Given a covering clutter, a *magic permutation* (with respect to the covering clutter) is a permutation of $[d]$ that is realizable by a *positive* point $x \in P$. Obviously, x can be chosen to be rational if it exists at all. In the cubical situation we are discussing here, all points in P° are positive, so magic permutations are identical to P-realizable and therefore to s-realizable permutations. A permutation σ of $[d]$ defines a *reverse dominance order* on the power set $\mathcal{P}([d])$ by

> $L \preccurlyeq_\sigma L'$ if, when L and L' are written in decreasing order according to σ, say $L = \{\sigma j_1, \ldots, \sigma j_l\}$ where $j_1 > \cdots > j_l$ and $L' = \{\sigma j'_1, \ldots, \sigma j'_{l'}\}$ where $j'_1 > \cdots > j'_{l'}$, then $l \leq l'$ and $j_1 \leq j'_1, \ldots, j_l \leq j'_l$.

This is a partial order on $\mathcal{P}([d])$.

Conjecture 3.7 (Magic permutations). A permutation σ of $[d]$ is realizable by a positive point in the magic subspace s of the covering clutter $([d], \mathcal{L})$ if and only if $\mathcal{L}$ is an antichain in the reverse dominance order due to σ.

The archetype is magic squares, where a magic permutation is a permutation of the cells of the square that is obtained from some magic labelling by arranging the cells in increasing order. We have verified the conjecture for 3×3 magic and semimagic squares (Examples 3.11 and 3.12).

A permutation being a total order on $[d]$, one could generalize to total preorders, in which the antisymmetric law is not required. We propose that the conjecture remains true when permutations are replaced by total preorders.

The necessity for $\mathcal{L}$ to be an antichain is obvious. It is the same with the extension to linear forms. Such a form is *positive* if all its coefficients are positive numbers. We define the *reverse dominance order due to* σ on the set of positive linear forms on $\mathbb{R}^d$ by

> $f \preccurlyeq_\sigma f'$ if, writing $f = \sum_{k=1}^d a_k x_k$ and $f' = \sum_{k=1}^d a'_k x_k$, then
> $$\sum_{k=j}^{d} a_{\sigma k} \leq \sum_{k=j}^{d} a'_{\sigma k} \text{ for every } j = 1, 2, \ldots, d.$$

Conjecture 3.8 (Magic permutations for linear forms). Let $f_1, \ldots, f_m$ be positive forms on $\mathbb{R}^d$ and s the subspace on which they are all zero. A permutation σ of $[d]$ is realizable by a positive point in s if and only if $\{f_1, \ldots, f_m\}$ is an antichain in the reverse dominance order of forms.

Example 3.9 (Cubical symmetry). A *cubically symmetric* magic or semimagic square has the property that any two cells that lie opposite each other across the center have sum equal to t, the cell-value bound, and the center cell (if there is one) contains the value $\frac{1}{2}t$. This definition is generalized from that of *associated* square in [1] (*symmetrical* square in [2]), in which the entries are $1, 2, \ldots, n^2$ and a symmetrical pair sums to n^2+1. (Cubical symmetry contrasts with affine symmetry, which we shall treat shortly.) The novel feature is the additional linear restraints besides the magic sum conditions: these are $y_{ij} + y_{n+1-i,n+1-j} = t$ for all $i, j \in [m]$. Translated into the language of $(1/t)$-fractional vectors $x = t^{-1}y \in t^{-1}\mathbb{Z}^{n^2}$, we require

$$x_{ij} + x_{n+1-i,n+1-j} = 1. \tag{3.2}$$

The effect is to reduce the magic subspace s to a smaller subspace s', but Theorems 3.1 and 3.5 apply (with s' replacing s). In the next lemma we generalize symmetry to fairly arbitrary covering clutters on the point set $[n]^2$. The definition is the same: we take the magic sum conditions and the symmetry equations (3.2). The magic subspace $s \subseteq \mathbb{R}^{n^2}$ is defined by equality of line sums; the *cubically symmetric magic subspace*, s', is the affine subspace of s in which (3.2) is valid; then $P = [0,1]^{n^2} \cap s'$ and $\mathcal{H} = \mathcal{H}[K_{n^2}]^{s'}$.

Lemma 3.10. *If $([n]^2, \mathcal{L})$ is a covering clutter of constant line size such that s' is not contained in a hyperplane $x_{ij} = x_{kl}$ of $\mathcal{H}[K_{n^2}]$, and in particular if a cubically symmetric, strongly magic labelling exists, then P and $\mathcal{H}$ are transverse.*

Proof. Since $\frac{1}{2}\mathbf{1} \in P \cap \bigcap \mathcal{H}$, every flat of $\mathcal{H}$ intersects P°. ∎

The hypothesis on s' is satisfied in the case of cubically symmetric magic squares of side $n \geq 3$ because such squares are known to exist, using the values $1, 2, \ldots, n^2$ if $n \not\equiv 2 \mod 4$; in fact, cubically symmetric pandiagonal squares exist if $n \geq 4$. See [2, pp. 204–211].

Example 3.11 (Magic squares of order 3: cubical count). Define $M_\mathrm{c}^\circ(t)$ for $t = 1, 2, 3, \ldots$ to be the number of 3×3 magic squares in which each cell value is less than t. From [5], the Ehrhart quasipolynomial is

$$M_\mathrm{c}^\circ(t) = \begin{cases} \frac{t^3-16t^2+76t-96}{6} = \frac{(t-2)(t-6)(t-8)}{6} & \text{if } t \equiv 0,2,6,8 \mod 12, \\ \frac{t^3-16t^2+73t-58}{6} = \frac{(t-1)(t^2-15t+58)}{6} & \text{if } t \equiv 1 \mod 12, \\ \frac{t^3-16t^2+73t-102}{6} = \frac{(t-3)(t^2-13t+34)}{6} & \text{if } t \equiv 3,11 \mod 12, \\ \frac{t^3-16t^2+76t-112}{6} = \frac{(t-4)(t^2-12t+28)}{6} & \text{if } t \equiv 4,10 \mod 12, \\ \frac{t^3-16t^2+73t-90}{6} = \frac{(t-2)(t-5)(t-9)}{6} & \text{if } t \equiv 5,9 \mod 12, \\ \frac{t^3-16t^2+73t-70}{6} = \frac{(t-7)(t^2-9t+10)}{6} & \text{if } t \equiv 7 \mod 12. \end{cases}$$

The constant term of $M_\mathrm{c}(t) = (-1)^3 M_\mathrm{c}^\circ(-t)$ is the number of magic permutations, which is 16. These permutations are the rotations and reflections of the patterns

$$\text{(a)}\ \begin{array}{|c|c|c|}\hline 4&9&2\\\hline 3&5&7\\\hline 8&1&6\\\hline\end{array} \qquad\qquad \text{(b)}\ \begin{array}{|c|c|c|}\hline 3&9&2\\\hline 4&5&6\\\hline 8&1&7\\\hline\end{array}$$

In these diagrams the numbers are not cell values but rather permutation positions: the largest value is in the cell marked 9, the next largest in that marked 8, and so on. (To realize the permutations by magic squares, (a) can be left untouched but (b) needs numbers.) The general form of a magic square is, up to the eight symmetries and an additive constant on each value,

$-\beta$	$\alpha+\beta$	$-\alpha$
$-(\alpha-\beta)$	0	$\alpha-\beta$
α	$-(\alpha+\beta)$	β

where $\alpha > \beta > 0$ and $\alpha \neq 2\beta$. If $\alpha > 2\beta$ we get the magic permutation (a); if $\alpha < 2\beta$ we get (b). This proves there are just 16 magic permutations.

Example 3.12 (Semimagic squares of order 3: cubical count). Let $S_c^\circ(t)$, for $t > 0$, be the number of semimagic squares of order 3 in which every entry is less than t. [5] has exact formulas. The constant term $|S_c(0)| = 1296 = 6^4$ equals the number of 3×3 *semimagic permutations*, that is, magic permutations for semimagic squares of order 3, in agreement with Conjecture 3.7. We verified this by hand, finding all semimagic permutations of order 3, based on the fact that a normalized semimagic permutation is a linear extension of the partial ordering implied by the supernormalized form of a semimagic square developed in [5].

3.3. **Affine magic.** The affine approach counts magic squares, and magic labellings in general, by the magic sum. In the general situation the magic subspace s is defined by a rational, nonhomogeneous linear system

$$f_1(x) = f_2(x) = \cdots = f_m(x) = 1 \tag{3.3}$$

that we assume is consistent. The magic polytope P is the nonnegative part of s, that is,

$$P := s \cap O,$$

where $O := \mathbb{R}^d_{\geq 0}$, the nonnegative orthant, and the hyperplane arrangement is $\mathcal{H} := \mathcal{H}[K_d]^s$.

Affine magic is quite similar to cubical magic, but there is something new: one has to worry about boundedness of P. Obviously, P is bounded if the defining linear forms f_i in (3.3) are positive and every variable appears in a form. If the latter fails, we are simply in the wrong dimension, so we make the overall assumption in this section that *every variable appears in at least one form.* (A covering clutter satisfies this automatically.) As in the cubical treatment, one must make sure that P affinely spans s (or else change s in the theorems to $\operatorname{aff} P$) and one has to be concerned about transversality of P and $\mathcal{H}$.

Lemma 3.13. *If P is not contained within a coordinate hyperplane, it spans s.*

Proof. The proof is similar to that of Lemma 3.3. First, P contains a point of the open positive orthant O°. By the hypothesis P contains points x^i (not necessarily distinct) with $0 < (x^i)_i \leq 1$. The barycenter x of these points lies in O° and in P.

Therefore P does not lie in the union of the coordinate hyperplanes. Since P is bounded by the coordinate hyperplanes, $P^\circ = O^\circ \cap s$, which spans s. ∎

Let $s \subseteq \mathbb{R}^d$ be the solution space of (3.3), where the f_i are rational linear forms, and let $P := s \cap O$. For $t = 1, 2, \ldots$, let

$A^\circ(t) :=$ the number of integer points $x \in tP$ with distinct positive entries,

and let

$A(t) :=$ the number of pairs (x,σ) consisting of a nonnegative integer point $x \in tP$ and a compatible P°-realizable permutation σ of $[d]$.

When P is bounded and spans s, Theorem 3.1 applies.

Theorem 3.14 (Magic enumeration by line sums)**.** *Suppose that P is bounded and does not lie within a coordinate hyperplane. Then A° and A are quasipolynomials with leading term $(\operatorname{vol} P)t^{\dim s}$ and with constant term $A(0)$ equal to the number of permutations of $[d]$ that are realizable in s. Furthermore, $(-1)^{\dim s}A^\circ(-t) = A(t)$.*

Proof. The proof is similar to that of Theorem 3.4. Theorem 3.1 shows that $A = E_{P,\mathcal{H}}$ and $A^\circ = E^\circ_{P^\circ,\mathcal{H}}$ are reciprocal quasipolynomials. The constant term equals the number of permutations that are realizable in P. To prove these are the same permutations that are realizable in P°, we show that a permutation realized in P is realized by a point in P°.

Any permutation σ that is realized in P is realized by a region R of $\mathcal{H}[K_d]^s$. Since R is relatively open in s, $R \cap P^\circ$ is nonempty; and it realizes σ. ■

We want to know when P and $\mathcal{H}$ are transverse so that the Möbius-function formulas of Theorem 2.2 will apply.

Lemma 3.15. *Assume P is bounded. P and $\mathcal{H}$ are transverse if s is not contained in any hyperplane $x_j = x_k$ and all forms f_i have equal positive weight.*

Proof. We need to verify that for no $u \in \mathcal{L}(\mathcal{H})$ do we have $\varnothing \neq P \cap u \subseteq \partial P$. But $P\cap u = P\cap v$ for some $v \in \mathcal{L}(\mathcal{H}[K_d])$, and ∂P is contained in the union of the coordinate hyperplanes since they determine the facets of P. If some $v \cap P$ lies in a coordinate hyperplane, then the same is true of $v = \langle \mathbf{1} \rangle$; but $\langle \mathbf{1} \rangle \cap P$ is in a coordinate hyperplane if and only if it is $\varnothing$ or $\{0\}$. These are impossible with forms as in (3.3). If the forms have equal weight $c > 0$, then $\langle \mathbf{1} \rangle \cap P = \{c^{-1}\mathbf{1}\}$. ■

Theorem 3.16. *With s defined by forms of constant positive weight, and assuming P is bounded and a magic labelling exists, we have*

$$A^\circ(t) = \sum_{u \in \mathcal{L}(P^\circ,\mathcal{H})} \mu(\hat{0},u) E_{u\cap O^\circ}(t) \ ,$$
$$A(t) = \sum_{u \in \mathcal{L}(P^\circ,\mathcal{H})} |\mu(\hat{0},u)| E_{u\cap O}(t) \ ,$$

where μ is the Möbius function of $\mathcal{L}(P^\circ,\mathcal{H})$.

Proof. Transversality holds by Lemma 3.15, since a magic labelling exists if and only if s does not lie in any hyperplane $x_j = x_k$. Apply Theorem 2.2 in s. ■

Some interesting examples are the affine versions of the cubical examples we already mentioned. In many of them the affine intersection poset is combinatorially equivalent to the cubical intersection poset. Let P_{c} be the cubical polytope $s \cap (0,1)^d$; P is still the affine polytope $s \cap O$.

Lemma 3.17. *Suppose that s is defined by positive forms such that each variable x_j has some form in which its coefficient is at least 1. Assume also that a magic labelling exists. Then the affine and cubical intersection posets are naturally isomorphic: $\mathcal{L}(P_{\mathrm{c}}^\circ,\mathcal{H}[K_d]) \cong \mathcal{L}(P^\circ,\mathcal{H}[K_d])$ by $u \cap P_{\mathrm{c}}^\circ \mapsto u \cap P^\circ$ for $u \in \mathcal{L}(\mathcal{H}[K_d])$ such that $u \cap P_{\mathrm{c}} \neq \varnothing$.*

Proof. The assumption on the coefficients implies that $P = P_c \cap s$. ■

Example 3.18 (Lines of constant length, cf. Example 3.6). The linear equations that express the existence of a magic sum t take the form

$$\sum_{j \in L} x_j = t \text{ for all } L \in \mathcal{L} \ .$$

Theorems 3.14 and 3.16 and Lemma 3.17 all apply as long as a magic labelling exists; thus, $A^\circ(t)$ and $A(t)$ are reciprocal quasipolynomials in the magic sum t, etc.

For general covering clutters Theorem 3.14 applies, showing that $A(t)$ and $A^\circ(t)$ are reciprocal quasipolynomials in t—provided that magic labellings exist at all. With that assumption, Lemma 3.17 also applies. The affine magic subspace, call it s_1, defined by $\sum_{i \in L} x_i = 1$ for all $L \in \mathcal{L}$, is an affine subspace of the homogeneous magic subspace, call it s_0, and s_0 is the linear subspace generated by s_1. This means that the permutations realizable in s_0 and s_1 are the same, so all our comments on magic permutations with respect to a covering clutter, in the context of cubical counting, apply as well to affine enumeration, except that we do not in general know that all s-realizable permutations are P-realizable.

Example 3.19 (Affine symmetry; cf. Example 3.9). An *affinely symmetric* magic or semimagic square has the property that the average value of any two cells that lie opposite each other across the center equals the average cell value, t/n, and the center cell (if n is odd sothere is one) contains the value t/n. (Of course, one cannot expect such squares to exist unless $t \equiv 0 \pmod{n}$ for odd n or $t \equiv 0 \pmod{n/2}$ for even n.) This definition is another generalization of that of associated square. The additional linear restraints are $y_{ij} + y_{n+1-i,n+1-j} = 2t/n$ for all $i, j \in [n]$. Translated into the language of $(1/t)$-fractional vectors $x = t^{-1}y \in t^{-1}\mathbb{Z}^{n^2}$, we require

$$x_{ij} + x_{n+1-i,n+1-j} = \frac{2}{n} \ . \tag{3.4}$$

The effect is to reduce the magic subspace s to a smaller subspace s'. Theorems 3.1 and 3.16 apply, with s' replacing s, by the following lemma, in which we generalize symmetry to covering clutters on $X = [n]^2$, with magic sum conditions and the symmetry equations (3.4). The magic subspace $s \subseteq \mathbb{R}^{n^2}$ is defined by equality of line sums; the *affinely symmetric magic subspace*, s', is the affine subspace of s in which (3.2) is valid; then $P = \mathbb{R}^{n^2}_{\geq 0} \cap s'$ and $\mathcal{H} = \mathcal{H}[K_{n^2}]^{s'}$.

Lemma 3.20. *If $([n]^2, \mathcal{L})$ is a covering clutter with lines of size n such that s' is not contained in a hyperplane $x_{ij} = x_{kl}$ of $\mathcal{H}[K_{n^2}]$, and in particular if an affinely symmetric magic labelling exists, then P and $\mathcal{H}$ are transverse.*

Proof. Since $n^{-1}\mathbf{1} \in P \cap \bigcap \mathcal{H}$, every flat of $\mathcal{H}$ intersects P°. ■

Example 3.21 (Magic squares of order 3: affine count). Let $M_{\mathrm{a}}^{\circ}(t)$, for $t > 0$, be the number of magic 3×3 squares with magic sum t. In [5] we find that

$$M_{\mathrm{a}}^{\circ}(t) = \begin{cases} \frac{2t^2-32t+144}{9} = \frac{2}{9}(t^2-16t+72) & \text{if } t \equiv 0 \mod 18, \\ \frac{2t^2-32t+78}{9} = \frac{2}{9}(t-3)(t-13) & \text{if } t \equiv 3 \mod 18, \\ \frac{2t^2-32t+120}{9} = \frac{2}{9}(t-6)(t-10) & \text{if } t \equiv 6 \mod 18, \\ \frac{2t^2-32t+126}{9} = \frac{2}{9}(t-7)(t-9) & \text{if } t \equiv 9 \mod 18, \\ \frac{2t^2-32t+96}{9} = \frac{2}{9}(t-4)(t-12) & \text{if } t \equiv 12 \mod 18, \\ \frac{2t^2-32t+102}{9} = \frac{2}{9}(t^2-16t+51) & \text{if } t \equiv 15 \mod 18, \\ 0 & \text{if } t \not\equiv 0 \mod 3. \end{cases}$$

Xin [17] has another way to find the generating function of $M_{\mathrm{a}}^{\circ}(t)$ (with the minor difference that he allows zero entries), by applying MacMahon's partition calculus.

The constant term $M_{\mathrm{a}}(0) = 16$ is the number of magic permutations; this is the same number as with cubically counted magic squares, $M_{\mathrm{c}}(0)$ in Example 3.11.

Example 3.22 (Semimagic squares of order 3: affine count). Complete results for this example are in [5].

4. Latin squares join in magically

The general picture that encompasses latin squares is that of a covering clutter $(X, \mathfrak{L})$, as in Section 3, with an integer labelling $x : X \to \mathbb{Z}$ subject to the requirement that

$$x(e) \neq x(f) \qquad \text{if } e \text{ and } f \text{ lie in a line.}$$

This is a *latin labelling* of $(X, \mathfrak{L})$. The graph of forbidden equalities is therefore

$$\Gamma_{\mathfrak{L}} := \bigcup_{L \in \mathfrak{L}} K_L,$$

K_L being the complete graph on the node set L. Every graph is equal to $\Gamma_{\mathfrak{L}}$ for some choice of covering clutter. An orientation of $\Gamma_{\mathfrak{L}}$ is *acyclic* if it has no directed cycles. An orientation and a node c-coloring $x : V \to [c]$ are *compatible* if $x_j \geq x_i$ whenever there is a $\Gamma_{\mathfrak{L}}$-edge oriented from i to j [15].

A crucial decision is how to restrict the symbols of the latin labelling. One may simply specify the number of symbols allowed, say k, and an arbitrary symbol set, let us say $[k]$. Then the number of latin labellings equals the chromatic polynomial $\chi_{\Gamma_{\mathfrak{L}}}(k)$, so we would have merely an application of graph coloring. In this article, taking a leaf from the guidebook of magic squares, we focus on approaches which give rise to *magilatin squares*: we impose a summation condition on the lines, which may be either homogeneous:

(i) Set all line sums equal to each other.

or affine:

(ii) Set all line sums equal to t.

This tactic is not so strange as it may appear. A magilatin square with homogeneous line-sum requirements and symbols restricted to the interval $[1,n]$ is just a latin square with symbol set $[n]$. So is a magilatin square with affine line-sum restrictions, line sum $t = \binom{n+1}{2}$, and positive symbols. We are assuming that the affine constraint (ii) is supplemented by a positivity assumption and that the homogeneous constraint (i) is supplemented by the requirement that the symbols be drawn from the set $[k]$ for some k. To treat Latin rectangles similarly, as a special case of magilatin rectangles (see the introduction for the definitions), requires a generalization of (i) to multiple covering clutters; see Section 4.1. In every case the hyperplane arrangement is $\mathcal{H} := \mathcal{H}[\Gamma_{\mathcal{L}}]^s$, where s is the subspace of $\mathbb{R}^X$ determined by the appropriate line sum conditions.

4.1. **Cubical latinity.** The cubical approach to latin labellings counts them by the largest allowed value; if t is the parameter, the labellings counted are those with $0 < x_i < t$. We assume a *multiple covering clutter*, $(X; \mathcal{L}_1, \ldots, \mathcal{L}_k)$, that is, there are $k \geq 1$ covering clutters $\mathcal{L}_1, \ldots, \mathcal{L}_k$; in our examples k will be 1 for squares and 2 for rectangles. The labelling must be such that the lines in each class have equal sums; but the sums in different classes are independent of each other. For a magilatin labelling the numbers in each line must be distinct. To count the labellings we take the subspace s of $\mathbb{R}^d$ in which all line sums within each covering clutter $\mathcal{L}_i$ are equal. The polytope is $P := [0,1]^d \cap s$. This is as in Section 3.2, so Lemma 3.3 applies to assure that P spans s. For $t = 1, 2, \ldots$, let

$L^\circ_{\mathrm{c}}(t) :=$ the number of latin labellings x with equal line sums within each covering clutter and with entries that satisfy $0 < x_i < t$,

and let

$L_{\mathrm{c}}(t) :=$ the number of pairs consisting of a latin labelling x, with equal line sums within each covering clutter and with $0 \leq x_i \leq t$, and a compatible acyclic orientation of $\Gamma_{\mathcal{L}}$.

Theorem 4.1 (Magilatin enumeration by upper bound). *Suppose $P := [0,1]^d \cap s$ does not lie within a coordinate hyperplane. Then L°_{c} and L_{c} are quasipolynomials with leading term $(\operatorname{vol} P)t^{\dim s}$ and with constant term $L_{\mathrm{c}}(0)$ equal to the number of acyclic orientations of $\Gamma_{\mathcal{L}}$ that are realizable in s. Furthermore, $(-1)^{\dim s} L^\circ_{\mathrm{c}}(-t) = L_{\mathrm{c}}(t)$.*

Proof. Theorem 2.1 shows that $L_{\mathrm{c}} = E_{P,\mathcal{H}}$ and $L^\circ_{\mathrm{c}} = E^\circ_{P^\circ,\mathcal{H}}$ are reciprocal quasipolynomials. The remainder of the proof is as in Theorem 3.4. ■

Theorem 4.2. *If all lines have the same size and a latin labelling exists, then we have all the formulas of Theorem 3.5 with L_{c} in place of B.*

Proof. Apply Theorem 2.2. ■

Example 4.3 (Small latin shapes, counted cubically). We calculated two very small magilatin examples: a square and a rectangle.

In the 2×2 magilatin square it must be that $x_{11} = x_{22} \neq x_{12} = x_{21}$. With cubical constraints, then,

$$L^\circ_{\mathrm{c}}(t) = (t-1)(t-2).$$

For comparison, the number of squares without the magilatin distinctness requirement is $(t-1)^2$.

The 2×3 magilatin rectangle is much more complicated. Calculation with Maple shows that the denominator of P is 6 and that of $(P, \mathcal{H})$ is 12. This permits us to calculate the quasipolynomial (by actual count of rectangles up to $t = 60$). Its period turns out to be 4.

$$L^\circ_{\mathrm{c}}(t) = \begin{cases} \frac{t^3-12t^2+41t-30}{4} = \frac{(t-1)(t-5)(t-6)}{4} & \text{if } t \equiv 1 \mod 4, \\ \frac{t^3-12t^2+41t-42}{4} = \frac{(t-2)(t-3)(t-7)}{4} & \text{if } t \equiv 3 \mod 4, \\ \frac{t^3-12t^2+44t-48}{4} = \frac{(t-2)(t-4)(t-6)}{4} & \text{if } t \text{ is even.} \end{cases}$$

For comparison, the number of rectangles without the distinctness requirement is

$$\begin{cases} \frac{t^3-3t^2+3t-1}{4} = \frac{(t-1)^3}{4} & \text{if } t \text{ is odd,} \\ \frac{t^3-3t^2+6t-4}{4} = \frac{(t-1)(t^2-2t+4)}{4} & \text{if } t \text{ is even.} \end{cases}$$

Problem 4.4. We do not know why the coefficients alternate in sign, what causes the smallness of the differences among the constituents, nor what makes the even constituents have a smaller period than the odd constituents.

Example 4.5 (Magilatin 3×3 squares, counted cubically). See [5] for the complete solution.

4.2. **Affine latinity.** For affine counting of latin labellings we take a covering clutter $(X, \mathcal{L})$. s is the subspace in which all line sums equal 1, and $P = s \cap O$, as in affine magic (Section 3.3). As there, we can apply Lemma 3.13 to conclude that P spans s in all interesting cases. One advantage over magic is that, because we consider only line sums and not general linear forms, P is certain to be bounded. For $t = 1, 2, \ldots$, let

$L^\circ_{\mathrm{a}}(t) :=$ the number of latin labellings of $(X, \mathcal{L})$ with positive entries and all line sums equal to t,

and let

$L_{\mathrm{a}}(t) :=$ the number of pairs consisting of a nonnegative latin labelling with all line sums equal to t and a compatible acyclic orientation of $\Gamma_{\mathcal{L}}$ that is realizable in P°.

Lemma 4.6. *P and $\mathcal{H}$ are transverse if a positive latin labelling exists.*

Proof. As with Lemma 3.15. The existence of a positive latin labelling implies that s is not contained in any hyperplane $x_j = x_k$ for which e_i and e_j are collinear and, for every $v \in \mathcal{L}(\mathcal{H}[\Gamma_{\mathcal{L}}])$, $v \cap P$ is not contained in a coordinate hyperplane. ■

Theorem 4.7 (Magilatin enumeration by magic line sum)**.** *Suppose that a positive latin labelling exists for some t. Then L°_{a} and L_{a} are quasipolynomials with leading term* $(\operatorname{vol} P)t^{\dim s}$ *and with constant term $L_{\mathrm{a}}(0)$ equal to the number of acyclic orientations of $\Gamma_{\mathcal{L}}$ that are realizable in P. Furthermore, $(-1)^{\dim s} L^\circ_{\mathrm{a}}(-t) = L_{\mathrm{a}}(t)$.*

Proof. By Theorem 3.1, adapted to the latin nonequalities, together with Lemma 4.6 to ensure transversality so that realizability in P is equivalent to realizability in P°. ■

If every line has the same size, then the acyclic orientations that are realizable in P° are the same as those realizable in s, because then $\langle \mathbf{1} \rangle$ intersects P°; see the discussion following Theorem 3.14.

Theorem 4.8. *Assuming a positive latin labelling exists and all lines have the same size, we have the formulas of Theorem 3.16 with L_{a} in place of A.*

Proof. Transversality holds by Lemma 4.6. Apply Theorem 2.2 in s. ■

Example 4.9 (Small magilatin squares, counted affinely). It is easy to see that in a 2×2 magilatin square with affine constraints,

$$L_{\mathrm{a}}^{\circ}(t) = \begin{cases} t-1 & \text{if } t \text{ is odd,} \\ t-2 & \text{if } t \text{ is even.} \end{cases}$$

The period of 2 equals the denominator of $(P, \mathcal{H})$. The number of positive squares without the magilatin distinctness requirement is $t-1$.

Example 4.10 (Magilatin 3×3 squares, counted affinely). For the number of these squares see [5].

5. Generalized exclusions

We concentrated our treatment on exclusions of magic and latin type: that is, where all values, or all collinear values, are unequal. Other kinds of exclusion are possible. We want to mention the very natural *complementation* restrictions. In cubical enumeration we call x_i and x_j *complementary* if $x_i + x_j = t$. If we forbid certain pairs of values to be complementary, we pass from graphs to signed graphs, since the rule $x_i + x_j \neq t$ corresponds to a negative edge $-ij$. A nonequality $x_i \neq x_j$ corresponds to a positive edge, $+ij$; thus ordinary edges are positive. The exact application of signed graphs involves translation and halving of the centrally symmetric polytope $[-1,1]^d$ to $[0,1]^d$, along with corresponding translation of the signed-graphic hyperplane arrangement, as explained in [4, Section 5]. Because signed-graphic hyperplanes, as translated, give half-integral vertices, we expect a counting quasipolynomial with nonequalities and noncomplementarities given by a signed graph to have twice the period of a counting quasipolynomial pertaining to a similar unsigned graph of nonequalities. This is necessarily vague; we intend only a suggestion for research that we invite readers to explore.

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