Matrices in the Theory of Signed Simple Graphs (Outline)

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This is an expository survey of the uses of matrices in the theory of simple graphs with signed edges.

A *signed simple graph* is a graph, without loops or parallel edges, in which every edge has been declared positive or negative. For many purposes the most significant thing about a signed graph is not the actual edge signs, but the sign of each circle (or 'cycle' or 'circuit'), which is the product of the signs of its edges. This fact is manifested in simple operations on the matrices I will present.

I treat three kinds of matrices of a signed graph, all of them direct generalizations of familiar matrices from ordinary, unsigned graph theory.

The first is the adjacency matrix. The adjacency matrix of an ordinary graph has 1 for adjacent vertices; that of a signed graph has $+1$ or -1, depending on the sign of the connecting edge. The adjacency matrix leads to questions about eigenvalues and strongly regular signed graphs.

The second matrix is the vertex-edge incidence matrix. There are two kinds of incidence matrix of a graph (without signs). The unoriented incidence matrix has two 1's in each column, corresponding to the endpoints of the edge whose column it is. The oriented incidence matrix has a $+1$ and a -1 in each column. For a signed graph, there are both kinds of columns, the former corresponding to a negative edge and the latter to a positive edge.

Finally, there is the Kirchhoff or Laplacian matrix. This is the adjacency matrix with signs reversed, and with the degrees of the vertices inserted in the diagonal. The Kirchhoff matrix equals the incidence matrix times its transpose. If we multiply in the other order, the transpose times the incidence matrix, we get the adjacency matrix of the line graph, but with 2's in the diagonal.

All this generalizes ordinary graph theory. Indeed, much of graph theory generalizes to signed graphs, while much—though not all—signed graph theory consists of generalizing facts about unsigned graphs.

As this is a survey, I will give very few proofs. As it is an outline, I will give few references; they will be added to the final paper.

I. Fundamentals of Signed Graphs

I.A. Definitions about Vectors and Matrices.

A vector of all 1's is denoted by $\mathbf{j}$. The matrix J consists of all 1's.

I.B. Definitions about Graphs.

A graph is $\Gamma = (V, E)$ with vertex set V and edge set E. All graphs will be undirected, finite, and *simple*: no loops or multiple edges. $n := |V|$ is the *order* of the graph. $c(\Gamma)$ is the number of connected components of Γ. Γ^c is the complement of Γ. Δ also denotes a graph of order n.

An edge with endpoints v, w may be written vw or e_{vw}. An edge with endpoints v_i, v_j may also be written e_{ij}.

A *walk* in Γ is a sequence $W = e_{01}e_{12}\cdots e_{l-1,l}$ of edges, where the second endpoint v_i of $e_{i-1,i}$ is the first endpoint of $e_{i,i+1}$. Its *length* is l. Vertices and edges may be repeated. W is *closed* if $v_0 = v_l$ and $l > 0$; otherwise it is *open*. A *trail* is a walk with no repeated edges. A *path* is an open walk with no repeated vertices or edges. A *closed path* is a closed walk with no repeated vertices or edges except that $v_0 = v_l$.

Important Subgraphs. A subgraph of Γ is *spanning* if it contains all the vertices of Γ. A *circle* (or circuit, cycle, polygon) is a 2-regular connected subgraph. A *theta graph* consists of three internally disjoint paths joining two vertices. A *pseudoforest* is a graph in which every component is a tree or a tree with one extra edge forming a circle. A *block* of Γ is a maximal 2-connected subgraph, or an isthmus or an isolated vertex. A *cut* is the set of edges between a vertex subset and its complement, except that the empty edge set is not considered a cut.

The *adjacency matrix* of Γ is the $n \times n$ matrix $A(\Gamma)$ in which $a_{ij} = 1$ if v_iv_j is an edge and 0 if not. The *Seidel adjacency matrix* of Γ is the $n \times n$ matrix $S(\Gamma)$ in which $s_{ij} = 0$ if $i = j$ and otherwise -1 if v_iv_j is an edge, 1 if it is not. I will have much more to say about the Seidel matrix.

I.C. **Definitions about Signed Graphs.**

A *signed graph* is $\Sigma = (|\Sigma|, \sigma)$, where $|\Sigma| = (V, E)$ is a graph, called the *underlying graph*, and $\sigma : E \to \{+, -\}$ is the *sign function* or *signature*. Often, we write $\Sigma = (\Gamma, \sigma)$ to mean that the underlying graph is Γ. E^+ and E^- are the sets of positive and negative edges. The spanning subgraphs of positive and negative edges are $\Sigma^+ = (V, E^+)$ and $\Sigma^- = (V, E^-)$; they are unsigned graphs.

Σ is *homogeneous* if all edges have the same sign, and *heterogeneous* otherwise. It is *all positive* or *all negative* if all edges are positive or negative, respectively.

Signed graphs Σ and Σ' are *isomorphic* if there is a graph isomorphism $f : |\Sigma| \to |\Sigma'|$ that preserves edge signs.

I.D. **Examples.**

1. $+\Gamma$ denotes Γ with all positive signs. $-\Gamma$ denotes Γ with all negative signs.
2. K_Δ denotes a complete graph K_n with vertex set $V = V(\Delta)$, whose edges are negative if they belong to Δ and positive otherwise. That is, $(K_\Delta)^- = \Delta$ and $(K_\Delta)^+ = \Delta^c$.

I.E. **Walks, Circles, and their Signs.**

The sign of a walk $W = e_1e_2\cdots e_l$ is the product of its edge signs:

$$\sigma(W) := \sigma(e_1)\sigma(e_2)\cdots\sigma(e_l).$$

Thus, a walk is either positive or negative, depending on whether it has an even or odd number of negative edges, counted with their multiplicity in W if there are repeated edges.

A circle, being the graph of a closed path, has a definite sign, either positive or negative. It is easy to see that the number of negative circles in any theta subgraph of Σ is either 0 or 2.

I.F. **Balance.**

Σ, or a subgraph or edge set, is called *balanced* if every circle in it is positive. (Physicists picturesquely say *satisfied* and *frustrated* for 'balanced' and 'unbalanced'.) $b(\Sigma)$ is the number of balanced components of Σ. For $S \subseteq E$, $b(S)$ is the number of connected components of (V, S) that are balanced.

A circle is balanced if and only if it is positive. A walk is called balanced when its underlying graph is balanced; thus, a positive walk may be balanced or unbalanced, and the same holds for a negative walk.

Σ is *antibalanced* if $-\Sigma$ is balanced, equivalently if all even circles are positive and all odd circles are negative. One way to get an antibalanced signed graph is to give negative signs to all edges of a graph.

Harary's Balance Theorem [6]: *Σ is balanced if and only if there is a bipartition of V into X and Y such that an edge is negative if and only if it has one endpoint in X and one in Y.* (X or Y may be empty.) In other words, Σ is balanced if and only if E^- is empty or a cut. When Σ is balanced, a *Harary bipartition* is any bipartition as in the theorem. It is unique if and only if Σ is connected. It was not so easy to prove this theorem at the time, but with switching it becomes simple; see below.

It is easy to see that Σ is balanced if and only if every block is balanced.

A *deletion set* in Σ is an edge set S such that $\Sigma \setminus S$ is balanced. A *negation set* is an edge set S such that negating the sign of every edge in S makes Σ balanced. **Theorem** (Harary): *S is a balancing set if and only if it is a deletion set.* Thus, negation is just as good as deletion for achieving balance. It is easy to see that no minimal deletion set can contain a cut.

I.G. **Switching.**

Switching Σ means reversing the signs of all edges between a vertex set X and its complement. X may be empty. We say X is *switched* in Σ. The switched graph is written Σ^X. *Vertex switching* means switching a single vertex.

Another version of switching, which is equivalent to the preceding, is in terms of a function $\theta : V \to \{+, -\}$, called a *switching function.*. *Switching Σ by θ* means changing σ to σ^θ defined by

$$\sigma^\theta(vw) := \theta(v)\sigma(vw)\theta(w).$$

The switched graph is written $\Sigma^\theta := (|\Sigma|, \sigma^\theta)$.

If Σ can be switched to become Σ', we say Σ and Σ' are *switching equivalent*. Switching equivalence is an equivalence relation on signatures of a fixed graph. An equivalence class is called a *switching class*. $[\Sigma]$ denotes the switching class of Σ.

If Σ' is isomorphic to a switching of Σ, we say Σ and Σ' are *switching isomorphic*. Switching isomorphism is an equivalence relation on all signed graphs. (Often, switching isomorphism is not distinguished from switching equivalence, but I prefer to separate the two concepts.)

It is not hard to prove that Σ is balanced if and only if it switches to an all-positive signature, and it is antibalanced if and only if it switches to an all-negative signature. (To prove the first statement one can assume Σ is connected. Take a spanning tree and switch it to be all positive. Σ is balanced if and only if there are no remaining negative edges. The second statement follows by negation, or it has a similar proof.) Properties preserved by switching are the signs of circles, and balance or imbalance of Σ and of any subgraph. With switching I can give a short **Proof of Harary's Balance Theorem**: If there is a

Harary bipartition, every circle has an even number of negative edges, so Σ is balanced. If Σ is balanced, switch it to have as few negative edges as possible. Since Σ is balanced, this number is 0. Letting X be the set of switched vertices, the Harary bipartition is $\{X, V \setminus X\}$.

I.H. **History.**

Signed graphs were invented by Harary (1953) to treat a question in social psychology [2]. König [7, Section X.3] almost invented signed graphs. He defined a graph with distinguished edge set, proved Harary's Balance Theorem, and had switching in the form of adding a cut, but did not think of labelling the edges by the 2-element group, which I regard as the crucial step. Signed graphs have been invented again and again in many different contexts—thus showing that they are a natural concept.

Another Definition. Some people define a 'signed graph' to be a pair (Γ, E^-) where $E^- \subseteq E$. They define the edge signature as $\sigma : E \to \mathbb{Z}_2$ where $\sigma(e) = 1$ if $e \in E^-$ and 0 if $e \notin E^-$. They also tend to call positive edges 'even' and negative edges 'odd' (which is confusing since, e.g., a circle can then be both even in parity and odd in sign at the same time).

II. The Adjacency Matrix

The notation for the adjacency matrix is $A(\Sigma)$.

II.A. **Definition and Elementary Properties.**

The adjacency matrix $A = A(\Sigma)$ is an $n \times n$ matrix in which $a_{ij} = \sigma(v_i v_j)$ (the sign of the edge $v_i v_j$) if v_i and v_j are adjacent, and 0 if they are not. Thus A is a symmetric matrix with entries $0, \pm 1$ and zero diagonal, and conversely, any such matrix is the adjacency matrix of a signed simple graph. The absolute value matrix, $|A(\Sigma)|$, equals $A(|\Sigma|)$, the adjacency matrix of the underlying unsigned graph.

Powers of A count walks in a signed way. Let $w^+_{ij}(l)$ be the number of positive walks from v_i to v_j (that is, the sign product of the edges in W is positive), and let $w^-_{ij}(l)$ be the number of negative walks. Then the ij entry of A^l is $w^+_{ij}(l) - w^-_{ij}(l)$.

Let us define Σ to be *regular* if both Σ^+ and Σ^- are regular graphs. (There is no currently accepted definition of regularity of a signed graph.) Then Σ is regular if and only if $A\mathbf{j} = r\mathbf{j}$ for some real number r, and in fact

$$r = d^{\pm}(\Sigma) := d(\Sigma^+) - d(\Sigma^-),$$

where d denotes the degree of any vertex.

II.B. **Examples.**

1. If Σ is complete, i.e., $|\Sigma| = K_n$, then $A(\Sigma)$ is the Seidel adjacency matrix of the negative subgraph Σ^-. That is, the Seidel matrix of a graph is the adjacency matrix of a signed complete graph. This fact inspired my work on adjacency matrices of signed graphs.
2. If Σ is complete bipartite, i.e., $|\Sigma| = K_{r,s}$, then $A(\Sigma) = \begin{pmatrix} O & B \\ B^{\mathrm{T}} & O \end{pmatrix}$ where $B = B(\Sigma)$ is an $r \times s$ matrix of $+1$'s and -1's. If Σ is bipartite but not necessarily complete bipartite, then $A(\Sigma)$ has a similar form but B may have 0's as well as $+1$'s and -1's.

II.C. **History.**

As far as I know, the first adjacency matrix of a signed graph was defined by the social psychologists Abelson and Rosenberg [1]. Their definition was, from the mathematical viewpoint, weird and interesting. They used, instead of numbers, the symbols u, p, n, a, standing for 'unrelated', 'positive', 'negative', and 'ambiguous' and meaning, respectively, that the two persons had no relationship, a positive or negative relationship, or a relationship with both positive and negative aspects. They defined multiplication rules that enabled them to take a Hadamard (i.e., componentwise) product of matrices, with which they were able to detect interesting mathematical properties.

The standard adjacency matrix, which is what I have defined, appeared soon after, but I'm not sure exactly when and where. Harary certainly used it early on.

II.D. **Switching.**

Switching has a simple effect on $A(\Sigma)$. Given $X \subseteq V$, switching by X negates both the rows and columns of vertices in X. Given a function $\theta : V \to \{+1, -1\}$, switching by θ negates both the rows and columns of vertices in $\theta^{-1}(-1)$.

In matrix terms, let $\mathrm{Diag}(\theta)$ be the diagonal $(0, \pm 1)$-matrix with $\theta(v_i)$ in the ith diagonal position. Then switching by θ conjugates $A(\Sigma)$ by $\mathrm{Diag}(\theta)$; that is,

$$A(\Sigma^\theta) = \mathrm{Diag}(\theta) A(\Sigma) \,\mathrm{Diag}(\theta).$$

If Σ is bipartite with bipartition $V = V_1 \cup V_2$, we can examine the effect of switching X on B. Say the rows are indexed by V_1 and the columns by V_2. Switching negates each row of B corresponding to a vertex in $X \cap V_1$ and each column corresponding to a vertex in $X \cap V_2$.

II.E. **Eigenvalues.**

An *eigenvalue* of Σ is an eigenvalue of its adjacency matrix. Just as with ordinary graphs, a regular signed graph has $d^\pm(\Sigma)$ as an eigenvalue. Other eigenvalues are problematic, but they may give information about Σ. I know of nothing that has been done on this except for strongly regular signed graphs.

II.F. **Very Strong Regularity.**

Seidel [9, 10] discovered that a strongly regular graph has a nice definition in terms of its Seidel adjacency matrix S, namely, that

$$S^2 - tS - kI = p(J - I) \text{ and } S\mathbf{j} = \rho_0 \mathbf{j}$$

for some constants t, k, p, ρ_0 (thus $\mathbf{j}$ is an eigenvector of S). Here $k = n - 1$; the other constants have combinatorial interpretations. The cases $p = 0$ and $p \neq 0$ behave differently.

Recalling that $S(\Delta)$ is the same matrix as $A(K_\Delta)$, I was led to the following definition: A signed graph is *very strongly regular* [15] if its adjacency matrix satisfies

$$A^2 - tA - kI = p\bar{A} \text{ and } A\mathbf{j} = \rho_0 \mathbf{j}$$

for some constants t, k, p, ρ_0. Here $\bar{A}$ is the adjacency matrix of the complement of $|\Sigma|$. The combinatorial interpretation of these parameters is:

a. $|\Sigma|$ is k-regular.
b. $\rho_0 = d^\pm(\Sigma)$, the net degree (which is an integer).
c. $t = t^+_{ij} - t^-_{ij}$ where t^+_{ij}, t^-_{ij} are the numbers of positive and negative triangles on edge e_{ij}.

d. $p = p^+_{ij} - p^-_{ij}$ for any pair v_i, v_j of nonadjacent vertices, where p^+_{ij}, p^-_{ij} are the numbers of positive and negative length-2 paths joining the vertices.
e. t and p are independent of the choices of adjacent or nonadjacent vertices.

The main problem is to classify very strongly regular signed graphs. I am currently working on this [15]. Various simplifications are helpful. For instance, $-\Sigma$, whose signature is $-\sigma$, behaves just like Σ except that t and ρ_0 are negated. Thus, one may assume that t is nonnegative, or that Σ is not all negative, when it is convenient to do so.

The most important factor in the classification is which of p and t are 0. As is usual in such problems, there are numerical restrictions. It is interesting that some of the types include kinds of matrices that have already been studied for many years. I will run down the possibilities.

a. Homogeneous signed graphs can be assumed (by negation) to be all positive. $+\Gamma$ is very strongly regular iff Γ is a strongly regular unsigned graph. For the rest of the cases I assume Σ is inhomogeneous.
b. If $p = 0$, the defining equations are $A^2 = kI$ and $A\mathbf{j} = \rho_0\mathbf{j}$. Eigenvalue arguments show there is a positive integer s such that $k = s(s + |t|)$ and $\rho_0 = s + (t + |t|)/2$.
   i. When $t = 0$, $\rho_0 = \sqrt{k}$. Thus, A is a symmetric weighing matrix with zero diagonal and in each row $\binom{\sqrt{k}+1}{2}$ entries equal to $+1$ and $\binom{\sqrt{k}}{2}$ entries equal to -1.
   ii. When $t \neq 0$ one can analyse further but I omit the details.
c. The most complicated cases are when $p \neq 0$. Σ must be connected, and there is a positive integer q such that $p(n - 1 - k) + k = q(q + |t|)$.
   i. When $t = 0$, then q equals the eigenvalue ρ_0.
   ii. The case $t \neq 0$ is more complicated; I omit further description.

The bipartite case is noteworthy. There $A = \begin{pmatrix} O & B \\ B^{\mathrm{T}} & O \end{pmatrix}$ so the defining equations imply $BB^{\mathrm{T}} = kI$, $t = 0$, and $\rho_0 = \sqrt{k}$. $A\mathbf{j} = \rho_0\mathbf{j}$ means that B has line sums $\sqrt{k}$. Therefore, B is a square weighing matrix of order $n/2$ and weight k with all row and column sums $= \sqrt{k}$. If Σ is complete bipartite, B is a Hadamard matrix with constant line sums $\sqrt{n}$. Not all Hadamard matrices have this property, but infinitely many do.

III. Orientation

III.A. **Bidirected Graphs.**

In a *bidirected graph*, every edge has an independent orientation at each end. We think of these in two ways: as an arrow at each end, which may point towards or away from the endpoint, and as a sign $\eta(v, e)$ on the end of e that is incident with v, which is $+1$ if the arrow points to the endpoint, -1 if the arrow is directed away from the endpoint.

III.B. **Oriented Signed Graphs.**

A bidirected graph is naturally signed by the formula

$$\sigma(e) = -\eta(v, e)\eta(w, e)$$

for an edge e_{vw}. An edge is negative if its arrows both point toward their corresponding endpoints, or both away from their endpoints. An edge is positive if one arrow points at its endpoint while the other is directed away from its endpoint.

Conversely, an *orientation* of a signed graph Σ is a bidirection η of $|\Sigma|$ that satisfies the sign formula just given.

Reorienting an edge e_{vw} means replacing $\eta(v,e)$ and $\eta(w,e)$ by their negatives. In terms of arrows, it means reversing the arrows at both endpoints of the edge. This does not change the sign of the edge.

IV. The Incidence Matrix

The notation for an incidence matrix of a signed graph is $\mathrm{H}(\Sigma)$ (read 'Eta').

IV.A. **Definition.**

An *incidence matrix* of Σ is a $V \times E$ matrix in which the column of edge e has two entries ± 1, one in the row of each endpoint of e, and 0's elsewhere. The two nonzero entries must have product equal to $-\sigma(e)$; that is, they are equal if e is negative, but if e is positive, one is $+1$ and the other is -1.

The incidence matrix is not unique. The choice of signs in each column reflects a choice of orientation η of Σ: the (v,e) entry in $\mathrm{H}(\Sigma)$ is $\eta(v,e)$ if v is an endpoint of e and 0 if not. Conversely, the entries of an incidence matrix of Σ determine an orientation η.

The incidence matrix can be treated as a matrix over any field $\mathbf{k}$.

IV.B. **Rank.**

If $\mathbf{k}$ has characteristic 2, $\mathrm{H}(\Sigma)$ has rank $n - c(\Sigma)$. Otherwise, $\mathrm{H}(\Sigma)$ has rank $n - b(\Sigma)$.

IV.C. **The Kirchhoff Matrix and Matrix-Tree Theorems.**

The *Kirchhoff matrix* (also called the *Laplacian*) is

$$K := \mathrm{H}(\Sigma)\mathrm{H}(\Sigma)^{\mathrm{T}} = \Delta(|\Sigma|) - A(\Sigma),$$

where $\Delta(|\Sigma|)$ is the degree matrix of $|\Sigma|$, the diagonal matrix whose diagonal entries are the degrees of the vertices in the underlying graph. I state two signed-graphic analogs of the matrix-tree theorem. For proofs see the references; the first is not hard (it uses the Binet–Cauchy theorem in the standard way), but the second is rather complicated both to state fully and to prove.

a. The determinant $\det K$ is the sum, over all pseudoforests F with n edges and with no positive circles, of $4^{c(F)}$, where $c(F)$ is the number of components of F [12, Section 8A].
b. From K delete k rows, corresponding to vertices $r_1, \ldots, r_k$, and k columns, corresponding to $c_1, \ldots, c_k$, and then take the determinant. The resulting number is the sum of $\pm 4^q$, where q is the number of circles in F, over all $n-k$-edge spanning pseudoforests without positive circles such that each tree component of F contains exactly one r_i and one c_j [3]. The sign of the term is given by a complicated rule.

IV.D. **Examples.**

1. An *oriented incidence matrix* of the unsigned graph Γ is the same as an incidence matrix of the all-positive signed graph $+\Gamma$. Since $+\Gamma$ is balanced, the rank given by our formula equals $n - c(\Gamma)$, as is well known.

   The Kirchhoff matrix of $+\Gamma$ is simply that of Γ, i.e., $\Delta(\Gamma) - A(\Gamma)$. The fact that its determinant is zero is consistent with the general theorem on $\det K(\Sigma)$, because there are no negative circles in $+\Gamma$.

2. The incidence matrix of an all-negative signed graph, $-\Gamma$, can be defined to have $+1$'s in every nonzero position. This is the customary definition of the *unoriented incidence matrix* of Γ. Since an all-negative graph is balanced if and only if it is bipartite, the rank of the matrix (except in characteristic 2) equals $n - b$ where b is the number of bipartite components of Γ. This result was previously obtained by ad hoc methods (first by van Nuffelen [8], to my knowledge), but it is really a special case of the general rank theorem for signed graphs.

   The Kirchhoff matrix $K(-\Gamma)$ equals $\Delta(\Gamma) + A(\Gamma)$. Its determinant equals the sum of $4^{c(F)}$ over all n-edge pseudoforests F in which every component contains an odd circle.

V. Line Graphs

V.A. **Unsigned Line Graphs.**

The line graph of an unsigned graph Γ is denoted by $L(\Gamma)$. Its vertex set is $E(\Gamma)$; two edges are adjacent if they have a common endpoint in Γ. $L(\Gamma)$ has two kinds of distinguished circles: vertex triangles are formed by three edges incident with a common vertex, and derived circles are the line graphs of circles in Γ. Every circle in $L(\Gamma)$ is a set sum of vertex triangles and derived circles.

V.B. **Signed Line Graphs.**

The line graph $\Lambda(\Sigma)$ of Σ is a switching class, not a single signed graph. Its underlying graph is the line graph $L(|\Sigma|)$ of the underlying graph. To define $\Lambda(\Sigma)$ we may take the approach of edge orientation or a direct definition of the circle signs.

1. *Definition by Orientation.*

   Choose an orientation η of Σ. We define a bidirection η' of $L(|\Sigma|)$ and therefore a signature, thus forming the line signed graph $\Lambda(\Sigma)$. Two edges e_{vw}, e'_{vw} incident with a vertex v form an edge ee' in Λ, whose vertices are e_{vw} and e'_{vw}. An end of ee' may therefore be written (e_{vw}, e), corresponding to the end (v, e_{vw}) in Σ. Define $\eta'(e_{vw}, e) = \eta(v, e_{vw})$.

   In terms of arrows, bidirect each edge of Σ with two arrows as indicated by η, and let the arrow on (e_{vw}, e) point into the vertex e_{vw} iff the arrow on (v, e_{vw}) points into the vertex v in Σ.

   Reorienting an edge in Σ corresponds to switching the corresponding vertex in $\Lambda(\Sigma)$. Thus, $\Lambda(\Sigma)$ is well defined only up to switching. I.e., it is a well defined switching class.
2. *Definition by Circle Signs.*

   Make every vertex triangle negative and give to every derived circle the same sign as the circle in Σ it derives from. Other circles get signed by the following *sum rule*: If C is the set sum of certain vertex triangles and derived circles, its sign is the product of the signs of those vertex triangles and derived circles. One has to prove that the sum rule gives the same sign no matter how it is applied, which is done by showing that the definition by circle signs agrees with that by edge orientation, the latter being obviously well defined.

V.C. **Characterisation.**

Line graphs of signed simple graphs are characterisable by forbidden induced subgraphs. The exact subgraphs are unknown. They probably all have at most 6 vertices.

What is known is that the forbidden induced subgraphs for reduced line graphs of simply signed graphs without loops have at most 6 vertices. This takes us to signed graphs with parallel edges. Suppose Σ is allowed to have both positive and negative edges linking the same vertices. Then in the line graph $\Lambda(\Sigma)$, if there are edges $+ef$ and $-ef$ with the same endpoints but opposite signs, they are deleted. This gives the *reduced line graph* of Σ. If Σ has no parallel edges, no reduction is possible. Chawathe and Vijayakumar [4] found the 49 forbidden induced signed subgraphs (actually, switching classes) for reduced line graphs of signed graphs with parallel edges allowed (which they regard as the signed graphs 'representable by the root system D_∞').

V.D. **Examples.**

1. The line graph of the all-negative signed graph $-\Gamma$ is $[-L(\Gamma)]$, the switching class of the ordinary line graph with all negative signs.
2. The line graphs that are antibalanced are $[-\Delta]$ where Δ is an ordinary line graph or a Hoffman generalised line graph. Thus, the usual theory of line graphs is the case of all-negative signed graphs.

V.E. **Adjacency Matrix and Eigenvalues.**

The adjacency matrix is the $E \times E$ matrix given by

$$A(\Lambda(\Sigma)) = 2I - \mathrm{H}(\Sigma)^{\mathrm{T}}\mathrm{H}(\Sigma).$$

The largest eigenvalue of $A(\Lambda(\Sigma))$ is 2. It is a simple eigenvalue if Σ is connected. The connected signed graphs whose eigenvalues are at most 2 are the line graphs of signed graphs and a few sporadic examples. (This was found by Vijayakumar et al.)

VI. Other Aspects of Signed Graphs

Here are some topics that I omitted.

VI.A. **Spaces and Lattices.**

The row and null spaces of the incidence matrix tell a lot about the structure of Σ. The integral elements of the real row and null spaces present remarkable new phenomena, not seen with unsigned graphs [5].

VI.B. **Multigraphs.**

Most of what I've said works equally well when loops and multiple edges are allowed. Some of it becomes more complicated.

VI.C. **Matroids and Geometry.**

The incidence matrix of Σ is intimately connected with matroid theory and with the geometry of Coxeter hyperplane arrangements.

A second type of incidence matrix of a signed graph, the *augmented binary incidence matrix*, is the matrix over $\mathbb{F}_2$ whose rows are those of the ordinary incidence matrix of $|\Sigma|$ (oriented or not, it being the same over $\mathbb{F}_2$) together with one extra row in which the signs appear as 0 for a positive edge and 1 for a negative one. This matrix has been exploited effectively by Gerards and Schrijver, among others.

The adjacency matrix arises in the study of *semi-equiangular lines*, which are lines in $\mathbb{R}^d$ whose angles are either 90° or $\arccos\alpha$ where $0 < \alpha < 1$ [10]. If G is the Gram matrix of the lines, $\alpha^{-1}(G-I)$ is the adjacency matrix of a signed graph. This construction generalizes that of $A(\Lambda(\Sigma))$, where $\alpha = 1/\sqrt{2}$.

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