Matrices in the Theory of Signed Simple Graphs

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Signed graph:

A graph in which each edge has been labelled $+$ (positive) or $-$ (negative).

The Purpose of this Talk

To show some of the ways in which two simple matrices contribute to the theory of signed graphs.

Outline

Basic

Signed graphs.
The adjacency matrix.
The incidence matrix and orientation.
The Kirchhoff matrix and matrix-tree theorems.
The incidence matrix and the line graph.

Advanced

Very strong regularity.
Extensions of incidence matrices.
Matrices over the group ring.
Degree vectors.

A **signed graph**

$$\Sigma = (|\Sigma|, \sigma) = (V, E, \sigma)$$

consists of

- a graph $|\Sigma| = (V, E)$, called the **underlying graph**;
- a **sign function** (**signature**) $\sigma : E \to \{+, -\}$.

(F. Harary)

The **positive subgraph** and **negative subgraph** are the (unsigned) graphs

$$\Sigma^+ = (V, E^+) \text{ and } \Sigma^- = (V, E^-),$$

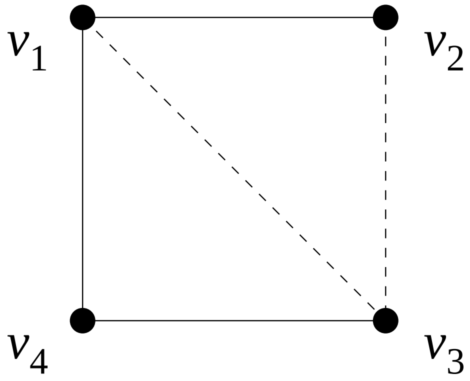
where E^+ and E^- are the sets of positive and negative edges.

Σ is **homogeneous** if its edges are all positive or all negative. It is *heterogeneous* otherwise. (M. Acharya)

Σ_1 and Σ_2 are **isomorphic** if there is an isomorphism of underlying graphs that preserves edge signs.

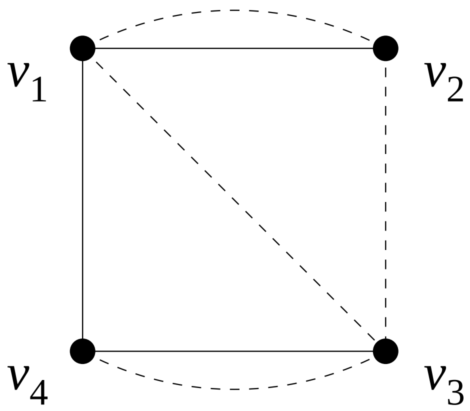
(a) Σ_4

A signed simple graph.



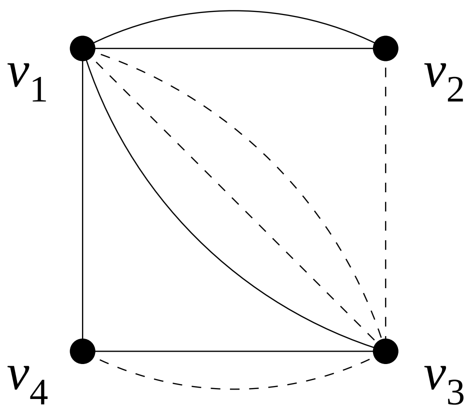
(b)

A simply signed graph.



(c)

A signed multigraph that is not simply signed.



Examples:

- Graph Σ_4, (a) in the figure.
  (Heterogeneous.)

- $+\Gamma$ denotes a graph Γ with all positive signs.
  (Homogeneous.)

- $-\Gamma$ denotes Γ with all negative signs.
  (Homogeneous.)

- K_Δ denotes a complete graph K_n, whose edges are negative if they belong to Δ and positive otherwise.
  (Homogeneous if $\Delta = K_n$ or K_n^c.
  Heterogeneous otherwise.)

- $-\Sigma := (V, E, -\sigma)$.
  (Occasionally, balance of $-\Sigma$ is important.)

A **walk** is a sequence of edges,

$$e_1 e_2 \cdots e_l,$$

whose edges are

$$e_1 = v_0 v_1,\ e_2 = v_1 v_2,\ \ldots,\ e_l = v_{l-1} v_l.$$

The vertices do not need to be distinct; also, the edges do not need to be distinct.

A **path** is a walk with no repeated vertices or edges.
A **closed path** is a walk with no repeated vertices or edges except that $v_0 = v_l$.
A **circle** ('circuit', 'cycle') is the graph of a closed path.

The **sign** of a walk $W = e_1 e_2 \cdots e_l$ is the product of its edge signs:

$$\sigma(W) := \sigma(e_1)\sigma(e_2)\cdots\sigma(e_l).$$

Thus, a walk is either **positive** or **negative**.

Fundamental fact about a signed graph:
The signs of the circles.

A subgraph or edge set is **balanced** if every circle in it is positive.

Theorem 1 (Harary's Balance Theorem)**.** *Σ is balanced $\iff$ there is a bipartition of V into X and Y such that an edge is negative if and only if it has one endpoint in X and one in Y. (X or Y may be empty.)*

$S \subseteq E(\Sigma)$:

- S is a **deletion set** if $\Sigma \setminus S$ is balanced.
- S is a **negation set** if negating S makes Σ balanced.

Theorem 2 (Harary's Negation-Deletion Theorem)**.** *An edge set is a minimal negation set $\iff$ it is a minimal deletion set.*
Every negation set is a deletion set; but not every deletion set is a negation set.

A **switching function** is $\theta : V \to \{+, -\}$. It changes the signs by the rule

$$\sigma^\theta(vw) := \theta(v)\sigma(vw)\theta(w).$$

Switching Σ by θ means replacing σ by σ^θ.
The switched graph is written

$$\Sigma^\theta := (|\Sigma|, \sigma^\theta).$$

Properties:

- Switching does not change signs of circles (easy).
- Switching does not change balance (easy).

Most of the important properties of signed graphs are invariant under switching!

Σ_1 and Σ_2 are **switching equivalent** if they have the same underlying graph and $\exists\ \theta : V \to \{+,-\}$ such that $\sigma_2 = \sigma_1^\theta$.
A switching equivalence class is called a **switching class**.

Σ_1 and Σ_2 are **switching isomorphic** if they have the same underlying graph and $\exists\ \theta : V \to \{+,-\}$ such that $\sigma_2 \cong \sigma_1^\theta$.
An equivalence class under switching isomorphism is called a **switching isomorphism class**, sometimes a 'switching class'.

Theorem 3 (Sozański; Zaslavsky)**.** *Σ_1 and Σ_2 are switching equivalent $\iff$ every circle has the same sign in both graphs.*
Σ_1 and Σ_2 are switching isomorphic $\iff$ there is an isomorphism of underlying graphs that preserves the signs of circles.

Switching gives short proofs of such results as Harary's balance theorem:

Σ is balanced $\iff$
$V = X \cup Y$ so that an edge is negative if and only if it has one endpoint in X and one in Y.

Proof of Harary's Balance Theorem:
If there is such a bipartition, then every circle has an even number of negative edges, so Σ is balanced.

If Σ is balanced, switch it to be all positive. (This is possible because all circles are positive.) Letting X be the set of switched vertices, the bipartition is $\{X,\ V \setminus X\}$.
$\square$

Adjacency matrix $A = A(\Sigma)$:

$$a_{ij} = \begin{cases} \sigma(v_i v_j) & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0 & \text{if they are not adjacent.} \end{cases}$$

Properties:

- Symmetric $(0, 1, -1)$-matrix with 0 diagonal.
- Every such matrix is the adjacency matrix of a signed simple graph.
- $|A| = A(|\Sigma|)$.
- Σ is regular $\iff$ $\mathbf{1}$ is an eigenvector of both $A(|\Sigma|)$ and $A(\Sigma)$.
- Rank(A): Known only if $|\Sigma|$ is regular.

Example:

$$A(\Sigma_4) = \begin{pmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Bidirected graph: each edge has two independent arrows, one at each end. Algebraically,

$$\eta(e,v) := \begin{cases} +1 \text{ if arrow into } v, \\ -1 \text{ if arrow out from } v. \end{cases}$$

Three kinds of edge:

- Both arrows are aligned: an ordinary **directed** edge.
- Both arrows point outwards: an **extraverted** edge.
- Both arrows point inwards: an **introverted** edge.

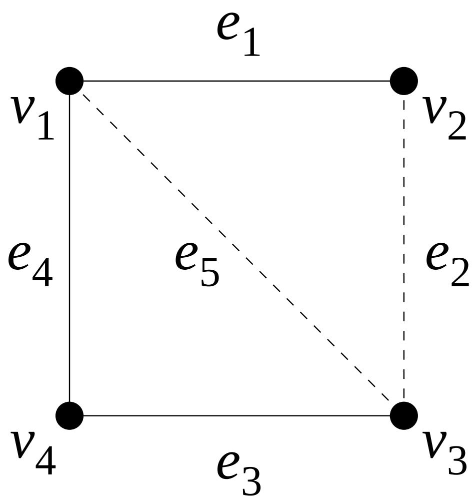
Bidirection implies edge signs:

(1) $\qquad \sigma(e) = -\eta(v,e)\eta(w,e)$ for an edge e_{vw}.

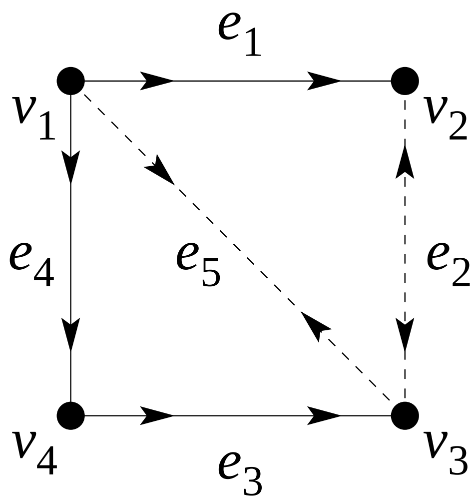
- Directed edge: *positive.*
- Extraverted: *negative.*
- Introverted: *negative.*

Orientation of Σ: a bidirection of $|\Sigma|$ whose signs obey the sign rule (1).

Directed graph = oriented $+\Gamma$.



Σ_4



A bidirected graph that is an orientation of Σ_4.

The incidence matrix of Σ_4 corresponding to the orientation η shown in the diagram:

$$\begin{array}{c} \\ \mathrm{H}(\Sigma_4, \eta) = \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & -1 & -1 \\ +1 & +1 & 0 & 0 & 0 \\ 0 & +1 & +1 & 0 & -1 \\ 0 & 0 & -1 & +1 & 0 \end{pmatrix} \end{array}$$

An **incidence matrix** of Σ is a $V \times E$ matrix

$$\mathrm{H}(\Sigma) = (\eta_{ve})_{v \in V,\, e \in E}$$

(read 'Eta') in which

- each column has two nonzero entries, which are ± 1, and
- the nonzero entries in the column of edge e_{uw} have product $\eta_{ue}\eta_{we} = -\sigma(e_{uw})$.

That is,

$$\begin{cases} \eta_{ue} = \eta_{we} & \text{if } e_{uw} \text{ is negative,} \\ \{\eta_{ue}, \eta_{we}\} = \{+1, -1\} & \text{if } e_{uw} \text{ is positive.} \end{cases}$$

Examples:

- **Unoriented incidence matrix of** Γ: two $+1$'s in each column. It is an incidence matrix $\mathrm{H}(-\Gamma)$.
- **Oriented incidence matrix of** Γ: an incidence matrix $\mathrm{H}(+\Gamma)$.

Properties:

- Rank $\mathrm{H}(\Sigma) = n - b(\Sigma)$, where $b(\Sigma) :=$ number of balanced components.
- Row space = cut space of Σ.
- Null space = cycle space of Σ.

Examples:

- rank $\mathrm{H}(-\Gamma) = n - b$ where $b :=$ number of bipartite components of Γ.
- rank $\mathrm{H}(+\Gamma) = n - c$ where $c :=$ number of components of Γ.
- rank $\mathrm{H}(\Sigma_4, \eta) = 4$ because Σ_4 has no balanced components. (It has one component, and that component contains negative circles.)
- **Theorem** (M. Doob). If Σ is all negative, then $\mathrm{Nul}(\mathrm{H}(\Sigma))$ is zero, or it contains a vector orthogonal to the all-1's row vector.

  Problem: Is there a generalisation to all signed graphs?

Extensions:

(1) *Augmented incidence matrix.*
We work over the 2-element field $\mathbb{Z}_2$. New notation: signs are $0, 1 \in \mathbb{Z}_2$.
Augmented binary incidence matrix: $\mathrm{H}(|\Sigma|)$ with an extra row containing the signs (as $0, 1$).
Combinatorial optimization, matroid theory. (Conforti, Cornuejols, et al.; Gerards & Schrijver)

(2) *Binet matrices.* (Appa et al.)
Signed-graph generalization of a network matrix.
We work over the reals. Remove redundant rows (if any) from $\mathrm{H}(\Sigma)$. Choose an invertible full-rank submatrix C and premultiply by C^{-1}. Remove the resulting I. This is a **binet matrix**.
Combinatorial optimization. Half-integral solutions.

(3) *Cycle, cut, flow, tension spaces and lattices.* (Chen et al.)
Combinatorial structure of the row space and null space of $\mathrm{H}(\Sigma)$ over the reals, rationals, or integers.

Kirchhoff matrix ('Laplacian matrix'):

$$K(\Sigma) = \Delta(|\Sigma|) - A(\Sigma),$$

where $\Delta(|\Sigma|) =$ degree matrix.

Properties:

- $K(\Sigma) = \mathrm{H}(\Sigma)\mathrm{H}(\Sigma)^{\mathrm{T}}$.
- Rank $K(\Sigma) = n - b(\Sigma)$.
- All eigenvalues ≥ 0.
- Multiplicity of 0 as an eigenvalue is $b(\Sigma)$.
- Eigenvalues tell us more about Σ. (Hou, Li, and Pan)
- Eigenvalues have interesting behavior when an edge is deleted. (Hou, Li, and Pan)
- If $|\Sigma|$ is k-regular: eigenvalues are those of $A(\Sigma)$, displaced by k.

The Kirchoff matrix of Σ_4:

$$K(\Sigma_4) = \Delta(|\Sigma_4|) - A(\Sigma_4) = \mathrm{H}(\Sigma_4, \eta)\mathrm{H}(\Sigma_4, \eta)^{\mathrm{T}}$$

$$= \begin{pmatrix} 3 & -1 & 1 & -1 \\ -1 & 2 & 1 & 0 \\ 1 & 1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

For comparison, the adjacency matrix:

$$A = \begin{pmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

and the degree matrix:

$$\Delta = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

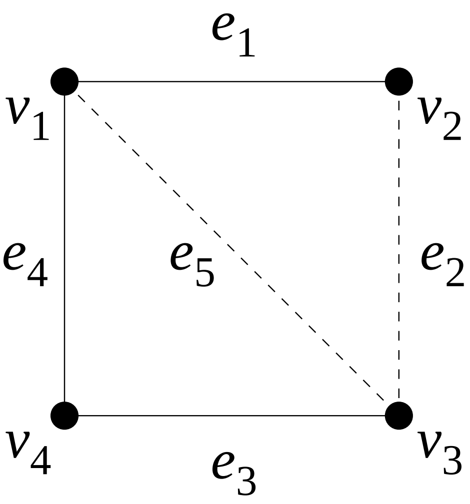
Line graph $\Lambda(\Sigma)$:
The line graph L of the underlying graph, with signs chosen in one of two equivalent ways.

(1) Orient Σ and use the orientation to get a bidirection of L. The bidirection implies signs on the edges of L. This is $\Lambda(\Sigma)$.

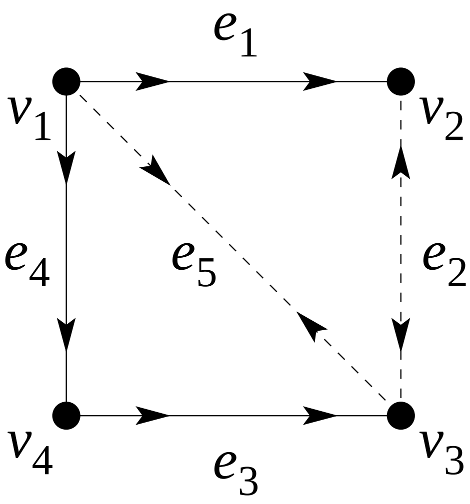
(2) Assign circle signs directly in L:
   (a) Each *vertex triangle* of L (formed by three edges incident with a common vertex) is negative.
   (b) Each *derived circle* (the sub–line graph of a circle C in Σ) gets the sign $\sigma(C)$ of the circle in Σ.
   (c) All other circle signs are determined from these.

Suppose Σ is a simply signed graph, but it has double edges of opposite sign (**negative digons**). Then the line graph also has negative digons. In A these edge pairs cancel to 0.
The **reduced line graph** $\bar{\Lambda}(\Sigma)$ is the line graph with negative digons deleted.

$$A(\bar{\Lambda}(\Sigma)) = A(\Lambda(\Sigma)).$$



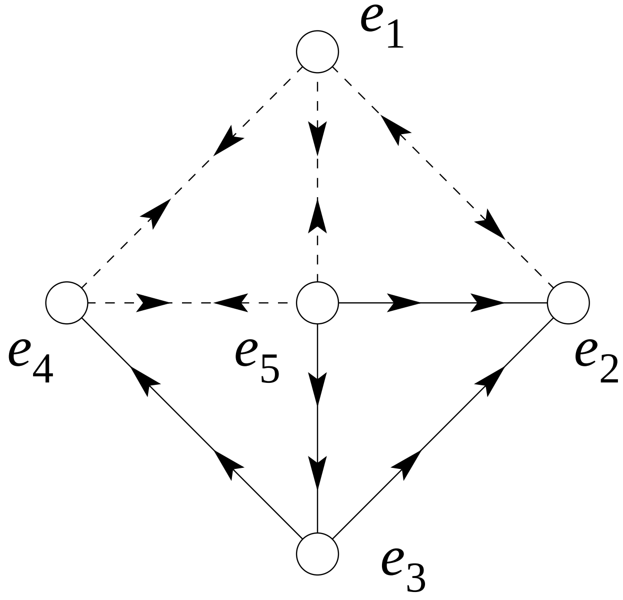
(a) Σ_4



(b) An orientation of Σ_4.

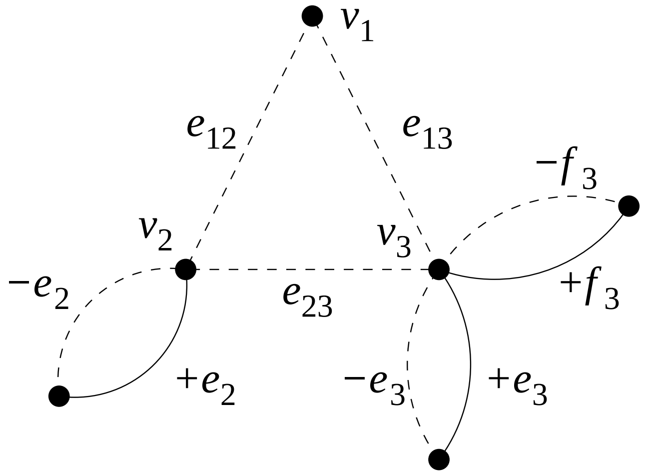
(c)

The bidirected graph that is the line graph of (b).



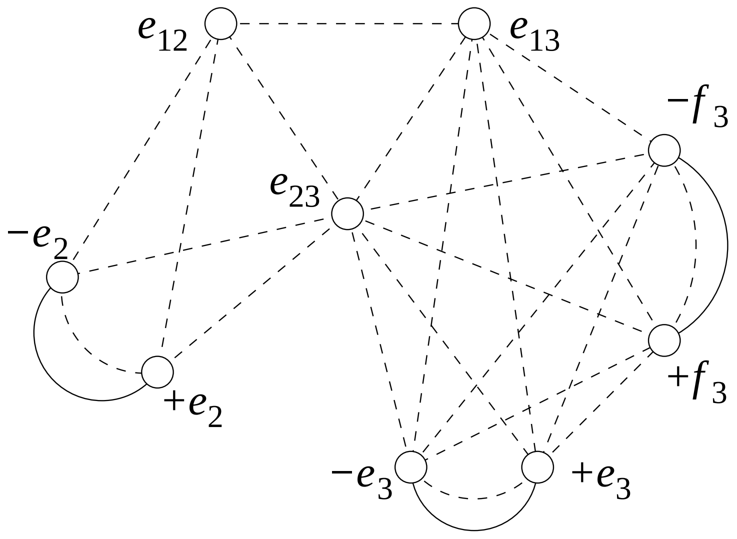
(a)

A signed graph.



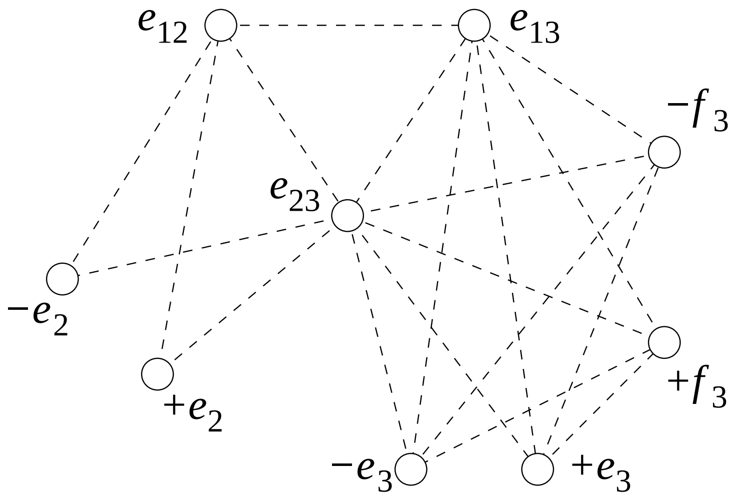
(b)

Its line graph.



(c)

Its reduced line graph.



Properties:

- The signs of $\Lambda(\Sigma)$ are only determined up to switching. The line graph is a *switching class*; one ought to write $[\Lambda(\Sigma)]$.
- $\Lambda(-\Gamma) = -L(\Gamma)$. Thus,
  ordinary graphs act like all-negative graphs in line graph theory.
- [For experts]
  $\Lambda(+\Gamma)^+ =$ the Harary–Norman line digraph of a digraph.
  (Orient $+\Gamma$, getting a digraph. The positive part of its line graph is the Harary–Norman line digraph.)
- Adjacency matrix in terms of the incidence matrix $\mathrm{H}(\Sigma)$:

$$A(\Lambda(\Sigma)) = 2I - \mathrm{H}(\Sigma)^{\mathrm{T}}\mathrm{H}(\Sigma).$$

  This has consequences. . .

Equation:

$$A(\Lambda(\Sigma)) = 2I - \mathrm{H}(\Sigma)^{\mathrm{T}}\mathrm{H}(\Sigma).$$

Eigenvalue Properties:

$*$ All eigenvalues of the line graph are ≤ 2.
The multiplicity of 2 is
$$|E(\Sigma)| - n + b(\Sigma)$$
because it = nullity of $\mathrm{H}(\Sigma)^{\mathrm{T}}$.

$*$ Ordinary line graphs $L(\Gamma)$ have all eigenvalues ≥ -2 $\leftrightarrow$ line signed graphs have all eigenvalues ≤ 2.

$*$ Hoffman's 'generalized line graph' is a reduced line graph of a signed graph.

$*$ *All signed graphs with the property that all eigenvalues are* ≤ 2 *are* reduced *line graphs*—with a few exceptions.
(Vijayakumar et al.: 'signed graphs represented by root system D_n.')

A graph Δ is **strongly regular** if

- it is regular (let $k =$ degree),
- any two adjacent vertices have exactly p common neighbors, and
- any two non-adjacent vertices have exactly q common neighbors.

Proposition 4 (Seidel's Matrix Definition)**.** *The graph Δ is strongly regular* $\iff$

$$A(K_\Delta)^2 - tA(K_\Delta) - \kappa I = \pi(J - I) \quad \textit{and} \quad A(K_\Delta)\mathbf{1} = \rho_0\mathbf{1},$$

where t, κ, π, ρ_0 are constants.

J is the all-1's matrix, $\kappa = n-1$, and t, π are determined by p, q.

Many results followed from this matrix method.

Problem:
Do something similar with signed graphs.

Σ is **very strongly regular** if

$$A^2 - tA - kI = p\bar{A} \text{ and } A\mathbf{1} = \rho_0\mathbf{1}$$

for some constants t, k, p, ρ_0. $\bar{A}$ is the adjacency matrix of the complement of $|\Sigma|$.

Combinatorial interpretation:

- $|\Sigma|$ is k-regular.
- $\rho_0 = d^{\pm}(\Sigma)$.
- $t = t_{ij}^+ - t_{ij}^-$ where $t_{ij}^+,\ t_{ij}^-$ = numbers of $+,\ -$ triangles on edge e_{ij}.
- $p = p_{ij}^+ - p_{ij}^-$ for $v_i \not\sim v_j$, where $p_{ij}^+,\ p_{ij}^-$ = numbers of $+, -$ paths of length 2 joining the vertices.

Problems:

- Classify very strongly regular signed graphs.
- Find a use for them.

Weighing matrix: $(0, \pm 1)$-matrix with $W^2 = wI$.

Hadamard ($w = n$), conference ($w = n - 1$), and all weighing matrices are adjacency matrices of very strongly regular bipartite graphs.

New notation:
Sign group $= \{\mathbf{p}, \mathbf{n}\}$ with $\mathbf{p} \leftrightarrow +1$, $\mathbf{n} \leftrightarrow -1$.

Group ring:

$$\mathbb{Z}\{\mathbf{p}, \mathbf{n}\} := \{a\mathbf{p} + b\mathbf{n} : a, b \in \mathbb{Z}\}$$

with the multiplication relations

$$\mathbf{p}^2 = \mathbf{n}^2 = \mathbf{p} \text{ and } \mathbf{pn} = \mathbf{np} = \mathbf{n}.$$

- **Group ring adjacency matrix** $\hat{A}(\Sigma)$:
$$\hat{a}_{ij} := a_{ij}^{+}\mathbf{p} + a_{ij}^{-}\mathbf{n},$$
where $a_{ij}^{+} :=$ number of $+v_iv_j$ edges, and $a_{ij}^{-} :=$ number of $-v_iv_j$ edges.
- **Group ring incidence matrix** $\hat{\mathrm{H}}(\Sigma)$:
$\mathrm{H}(\Sigma)$ with $+1 \mapsto \mathbf{p}$ and $-1 \mapsto \mathbf{n}$.
- **Group ring degree matrix**
$$\hat{\Delta}(|\Sigma|) := \Delta(|\Sigma|)\mathbf{p}.$$
- **Group ring Kirchhoff matrix**
$$\hat{K}(\Sigma) := \hat{\Delta}(|\Sigma|) - \hat{A}(\Sigma).$$

Examples:
Group-ring adjacency, incidence, and Kirchhoff matrices:

$$\hat{A}(\Sigma_4) = \begin{pmatrix} 0 & \mathbf{p} & \mathbf{n} & \mathbf{p} \\ \mathbf{p} & 0 & \mathbf{n} & 0 \\ \mathbf{n} & \mathbf{n} & 0 & \mathbf{p} \\ \mathbf{p} & 0 & \mathbf{p} & 0 \end{pmatrix}$$

$$\hat{\mathrm{H}}(\Sigma_4, \eta) = \begin{pmatrix} \mathbf{n} & 0 & 0 & \mathbf{n} & \mathbf{n} \\ \mathbf{p} & \mathbf{p} & 0 & 0 & 0 \\ 0 & \mathbf{p} & \mathbf{p} & 0 & \mathbf{n} \\ 0 & 0 & \mathbf{n} & \mathbf{p} & 0 \end{pmatrix}$$

$$\hat{K}(\Sigma_4) = \begin{pmatrix} 3\mathbf{p} & \mathbf{n} & \mathbf{p} & \mathbf{n} \\ \mathbf{n} & 2\mathbf{p} & \mathbf{p} & 0 \\ \mathbf{p} & \mathbf{p} & 3\mathbf{p} & \mathbf{n} \\ \mathbf{n} & 0 & \mathbf{n} & 2\mathbf{p} \end{pmatrix}$$

$w_{ij}^{+}(l) :=$ number of positive walks $v_i \to v_j$ of length l,
$w_{ij}^{-}(l) :=$ number of negative walks.

Theorem 5 (Signed Walks Count)**.** *$\left(\hat{A}^l\right)_{ij} = w_{ij}^{+}(l)\mathbf{p} + w_{ij}^{-}(l)\mathbf{n}$.*

Corollary 6 (Net Walks Count)**.** *$\left(A^l\right)_{ij} = w_{ij}^{+}(l) - w_{ij}^{-}(l)$.*

Corollary 6 follows from Theorem 5 using
The Natural Ring Homomorphism:

$$\mathbb{Z}\{\mathbf{p}, \mathbf{n}\} \to \mathbb{Z}$$
$$\alpha\mathbf{p} + \beta\mathbf{n} \mapsto \alpha + \beta.$$

Several kinds of 'signed' degree:

- **Positive degree** $d^+_\Sigma(v) =$ degree in Σ^+.
- **Negative degree** $d^-_\Sigma(v) =$ degree in Σ^-.
- **Net degree**
  $d^\pm_\Sigma(v) := d^+(v) - d^-(v) \in \mathbb{Z}$.
- **Degree pair**
  $\hat{d}_\Sigma(v) := d^+(v)\mathbf{p} + d^-(v)\mathbf{n} \in \mathbb{Z}\{\mathbf{p}, \mathbf{n}\}$.

Degree vector $\mathbf{d} := (d(v_1), d(v_2), \dots, d(v_n))^{\mathrm{T}}$, where $V = \{v_1, v_2, \dots, v_n\}$.

Formulas:
Let $\mathbf{1}$ be the all-1's vector. Then

$$\text{(a)}\ \ \mathbf{d}^\pm_\Sigma = A\mathbf{1}, \qquad \text{(b)}\ \ \hat{\mathbf{d}}_\Sigma = \hat{A}\mathbf{1}.$$

(b) is more informative.

(a) follows from (b) by applying the natural ring homomorphism.

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