Other Matroids from Graphs
(Outline)

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I. Many Kinds of Matroids

A. *Graphs*

1. Definitions about Graphs:
   a. $\Gamma = (V, E)$.
   b. $n := |V|$. (Usually finite.)
   c. Half and loose edges.
   d. Pseudoforests.
   e. Theta subgraphs, loose and tight handcuffs (minimal connected subgraphs with cyclomatic number 2 that are not theta graphs), loose and tight bracelets (minimal subgraphs with cyclomatic number 2 that are not theta graphs).
   f. $c(S)$ = number of connected components of (V, S).
2. Customary Matroids of Graphs:
   a. Graphic (circle, polygon, circuit, cycle) matroid $G(\Gamma)$.
   b. Cographic (bond, cocircuit, cocycle) matroid $G^*(\Gamma)$.

B. *Signed, Gain, and Biased Graphs*

1. What the Graphs Are:
   a. Definitions
      i. *Signed graph*: $\Sigma = (\Gamma, \sigma)$ where $\sigma : E \to \{+, -\}$.
      ii. *Gain graph*: $\Phi = (\Gamma, \varphi)$ with gain group $\mathfrak{G}$.
         (1) Integral gain graph: $\mathfrak{G} = \mathbb{Z}^+$ (additive group).
         (2) Real multiplicative gain graph: $\mathfrak{G} = \mathbb{R}^\times$.
         (3) Real additive gain graph: $\mathfrak{G} = \mathbb{R}^+$.
         (4) Modular gain graph: $\mathfrak{G} = \mathbb{Z}_m$.
      iii. *Biased graph*: $\Omega = (\Gamma, \mathcal{B})$ where $\mathcal{B}$ is a linear subclass of circles; that is, any theta subgraph contains 0, 1, or 3 but not exactly 2 circles of $\mathcal{B}$.
         (1) $\langle\Sigma\rangle = (\Gamma, \mathcal{B}(\Sigma))$, $\langle\Phi\rangle = (\Gamma, \mathcal{B}(\Phi))$: the associated biased graphs.
         (2) Sign-biased graphs are *additively biased*, i.e., a theta subgraph contains 1 or 3 circles of $\mathcal{B}$.
      iv. Balance
         (1) A subgraph or edge set is *balanced* if it contains no unbalanced circle or half edge.
         (2) A circle is balanced iff it belongs to $\mathcal{B}$.
         (3) An edge is unbalanced iff it is an unbalanced loop or a half edge.
         (4) $b(\Omega)$ = the number of balanced components of Ω. This applies to all subgraphs. In particular, $b(S) = b(V, S)$ for $S \subseteq E$.

(5) A *balancing set* is an edge set S whose deletion makes an unbalanced component balanced without increasing the number of components.
(6) A *balancing vertex* is a vertex whose deletion makes an unbalanced biased graph balanced. Existence makes Ω almost graphic.
(7) A biased graph is *entangled* if it is unbalanced, has no balancing vertex, and has no 2 disjoint negative circles.

v. Switching
(1) Switching function: $\tau : V \to \mathfrak{G}$.
(2) $\Phi^\tau := (\Gamma, \varphi^\tau)$ where $\varphi^\tau(e{:}vw) := \tau(v)^{-1}\varphi(e)\tau(w)$.
(3) Switching class: $[\Sigma]$ or $[\Phi]$.
(4) $\langle\Phi^\tau\rangle = \langle\Phi\rangle$.

b. Minors
i. Subgraphs and deletion ($\Omega \setminus S$ et al.) of edge set S: Obvious.
ii. Contraction Ω/S of edge set S: Vertex set is the set of vertex sets of balanced components of (V, S). Edge set is S^c.
iii. Contraction Φ/S of gain graph (and signed graph): Switch so balanced components of S have gains all equal to the identity. Then contract as in the previous, retaining the switched gains. (This is well defined up to switching.)

c. Examples
i. Graphic: $\varphi \equiv$ identity, or $\sigma \equiv +$ (all-positive signed graph).
ii. All-negative: $-\Gamma$. Positive circles are even circles. *Parity bias*: $\mathcal{B} = \{\text{even circles}\}$. An edge set is balanced iff it is bipartite.
iii. Contrabalanced graphs: $\mathcal{B} = \varnothing$, i.e., $\Omega = (\Gamma, \varnothing)$.
iv. Poise bias: For a mixed graph $\vec{\Gamma}$, $\vec{\mathcal{B}}$ is the class of circles having equally many edges in each direction (integral gains).
Modular poise bias: The same, but counted modulo m (gains in $\mathbb{Z}_m$).
v. Antidirection bias: For a digraph $\vec{\Gamma}$, $\mathcal{B}$ is the class of circles in which no two consecutive edges have the same direction around the circles.
vi. Full biased graph: $\Omega^\bullet$ has a half edge at each vertex.
vii. Group expansion $\mathfrak{G}\Delta$, where Δ is a simple graph: Each edge of Δ is replaced by one edge for each possible gain, i.e., for each group element.
viii. Full group expansion: $\mathfrak{G}\Delta^\bullet$, i.e., add a half edge to each vertex of the group expansion.

2. The Frame Matroid $G(\Omega)$:
a. Definitions
i. Ground set: E.
ii. Circuits: Balanced circles and contrabalanced handcuffs and theta graphs.
iii. Independent sets: Contrabalanced pseudoforests.
iv. Rank: $\mathrm{rk}(S) = |V| - b(S)$.

v. Cocircuits: Minimal subsets that increase $b(S)$. These include cut sets that cut off a balanced subset, minimal balancing sets, and combinations of cutsets and balancing sets.

b. Properties

i. Deletion and contraction are compatible with those in Ω.

ii. The lattice of closed sets (flats) is $\operatorname{Lat}\Omega$.

iii. The frame matroids of biased graphs are precisely the matroids called *frame matroids*, i.e., M is a submatroid of a matroid $M^\bullet$ that has a basis B such that every point of $M^\bullet$ is in a line determined by two basis elements.

iv. $G(\Omega)$ is essentially graphic, unlike $L(\Phi)$.

c. Examples

i. $G(+\Gamma)$ is the polygon matroid $G(\Gamma)$.

ii. $G(\Gamma, \varnothing)$ is the bicircular matroid of Γ. (Simões-Pereira, Klee) $\operatorname{Lat}(\Gamma^\bullet, \varnothing)$ is the geometric lattice of all forests of Γ. (Zaslavsky)

iii. $G(-\Gamma)$ is the even-circle matroid. (Tutte, Doob)

iv. $G(-K_5) = R_{10}$.

v. $G(\vec{\Gamma}, \vec{\mathcal{B}})$ where $\vec{\mathcal{B}}$ is poise or modular poise bias on a mixed graph. (Matthews)

vi. $G(\vec{\Gamma}, \mathcal{B})$ where $\mathcal{B}$ is antidirection bias on a digraph. (Matthews)

vii. $G(\mathfrak{G}K_n^\bullet)$ is the Dowling geometry of rank n of $\mathfrak{G}$.

viii. $G(\mathfrak{G}K_n)$ is the jointless Dowling geometry of rank n of $\mathfrak{G}$.

3. The Lift Matroid $L(\Omega)$:

a. Definitions

i. Ground set: E

ii. Circuits: Balanced circles and contrabalanced bracelets and theta graphs.

iii. Independent sets: Contrabalanced edge sets with no more than one balanced circle.

iv. Rank: $\operatorname{rk}_L(S) = |V| - c(S)$ if S is balanced, $|V| + 1 - c(S)$ if unbalanced.

v. The lattice of closed sets is $\operatorname{Lat} L(\Omega)$.

b. Properties

i. Deletion is compatible with that in Ω.

ii. Contraction of balanced edge sets is compatible with Ω.

iii. Contraction of an unbalanced set gives an unbiased graph.

iv. The lift matroids of biased graphs are precisely the elementary lifts of graphic matroids.

v. The lift theory applies to all matroids; not essentially graphic. (Dowling & Kelly)

c. Examples

i. $L(+\Gamma)$ is the polygon matroid $G(\Gamma)$.

ii. $L(\Gamma, \varnothing)$ is the bicircular lift matroid of Γ, the lift analog of the bicircular matroid.

iii. $L(-K_5) = R_{10} = G(-K_5)$.

iv. $L_0(\mathfrak{G}K_n)$ is the Dowling lift matroid of rank n of $\mathfrak{G}$, the lift analog of the Dowling matroid $G(\mathfrak{G}K_n^\bullet)$.

4. The Extended (or Complete) Lift Matroid $L_0(\Omega)$.
   a. Ground set: $E_0 = E \cup \{e_0\}$.
   b. The *extra point* e_0 counts as an unbalanced edge, but it has no endpoints. Given this interpretation, L_0 is L with the extra point.
   c. L_0 is a one-point extension of L, and L is an elementary lift of $G(\Gamma)$.
5. Comparison of Frame and Lift Matroids.
   a. Equal iff Ω has no 2 disjoint unbalanced circles.
   b. For an inseparable signed graph: (Zaslavsky; others)
      i. Equal iff regular.
      ii. If not equal, then $G(\Sigma)$ is nonbinary and $L(\Sigma)$ is binary but not regular.
6. The Balanced Semimatroid.
   a. Ground set: E.
   b. The sets of the semimatroid are the balanced sets of Ω.
   c. The rank function is the same as those of G and L.
   d. The closed sets are the balanced flats of G and L (which are the same).
   e. The semilattice of closed sets is $\mathrm{Lat}^{\mathrm{b}}\,\Omega$. It is a geometric semilattice (cf. Wachs & Walker).
7. Many Ways for Graph $\to$ Matroid. Summarizing, the (full) frame or (complete) lift matroid of a gain graph constructed from a graph Δ.
   a. $-\Delta$, $\mathfrak{G}\Delta$, $(\Delta, \varnothing)$, $\mathfrak{G}\Delta$, $\mathfrak{G}\Delta^\bullet$ as above.
   b. $(\mathfrak{G} \setminus 1)\Delta$, $1K_n \cup \mathfrak{G}\Delta$, $1\Delta \cup (\mathfrak{G} \setminus 1)K_n$, etc., etc. (Zaslavsky)
8. Common generalization of frame and lift matroids. (Whittle)

C. *Sublattices and Subposets*
1. Subposets of Dowling (semi)lattices (of sign group).
   a. Nested $\mathfrak{G}$-partitions. (Athanasiadis)
   b. Non-crossing (partial) $\mathfrak{G}$-partitions. (Blass & Sagan; Athanasiadis)
2. Essentially connected subgraphs of Dowling lattices (of sign group). (Björner & Sagan)

D. *Coxeter Matroids*
1. $\bigcup_\sigma G(\Gamma, \sigma)$ is a symplectic matroid. (T. Chow)

E. *Linearly Bounded ("Count") Matroids* (White & Whiteley)
1. Ground set: E.
2. Let the rank of an edge set S be $\le a|V(S)| + b$. Take the free-est matroid.
3. Examples: Circle matroid from $a = 1, b = -1$. Bicircular matroid from $a = 1,\ b = 0$.
4. Status: Not much is well understood, because great complications arise when $a > 1$ or $b > 0$.

F. *Matroidal Families* (Simões-Pereira, Schmidt)
1. Ground set: E.

2. Given: a family $\mathcal{C}$ of isomorphism types of finite graph. A subgraph (that is, its edge set) is a circuit iff the subgraph is in $\mathcal{C}$. $\mathcal{C}$ is a *matroidal family* if it produces a matroid for every graph.
3. Examples: Graphic matroid from circles. Bicircular matroid from handcuffs and theta graphs. Bicircular lift matroid from bracelets and theta graphs. Even-circle matroid from even circles and all handcuffs whose circles are odd. Linearly bounded (count) matroids.
4. Status: There are uncountably many, but very complicated, matroidal families. They seem to have no particular structure or properties.

G. *Delta Matroids*

1. Can be defined based on graphs in surfaces. (Bouchet)

II. Vector and Hyperplane Representations

A. *Multiplicative Gain Graphs*

Here the gain group is (contained in) $F^\times$ for a field or division ring.

1. The Incidence Matrix, $\mathrm{H}(\Phi)$ (read "Eta") $= (\eta_{ve})$.
   a. Gain graphs:
      i. $\mathrm{H}(\Phi)$ is a $V \times E$ matrix. The column of $e{:}vw$ is zero except that $\eta_{ve} = -1$ and $\eta_{we} = \varphi(e{:}vw)$. A balanced loop or loose edge has all zeros. An unbalanced loop or half edge at v has one nonzero entry η_{ve}.
      ii. Any column scaling also works.
      iii. $\mathrm{H}(\Phi)$ has rank $n - b(\Phi)$.
   b. Signed graphs:
      i. The incidence matrix is totally dyadic. It gives half-integral solutions of integral programs.
      ii. *Binet matrices* are a signed-graphic generalization of network matrices. (Appa & Kotnyek)
      iii. $\mathrm{H}(+\Gamma)$ is the usual oriented incidence matrix of Γ, which is totally unimiodular.
      iv. $\mathrm{H}(-\Gamma)$ is the usual unoriented incidence matrix of Γ. Its rank is n less the number of bipartite components of Γ. (van Nuffelen and many subsequent discoverers)
2. Vectors.
   a. The column of e is a vector $x_e \in F^n$.
   b. The linear dependences of vectors x_e are given by $G(\Phi)$.
3. Hyperplanes.
   a. The equations $x_j = x_i\varphi(e{:}v_iv_j)$ determine an arrangement of hyperplanes, $\mathcal{H}[\Phi]$, in F^n. This is the *two-term hyperplane representation* of Φ.
   b. The intersection flats of $\mathcal{H}[\Phi]$ correspond to the flats of $G(\Phi)$.
   c. Real or complex multiplicative gains: Is $\mathcal{H}[\Phi]$ supersolvable, inductively free, free?
4. Networks with Gains ("Generalized Networks"):
   a. Gain group $\mathbb{R}^\times_{>0}$, representing gains or losses in the edge.

b. Extensive literature on typical network-flow questions, going back 50 years.
c. Frame matroid and especially bases are fundamental, though usually implicit.

B. *Additive Gain Graphs*
Here the gain group is (contained in) F^+ for a field or division ring.

1. The Augmented Incidence Matrix.
   a. The incidence matrix is obtained by orienting Γ (arbitrarily), then adding to the incidence matrix $\mathrm{H}(\vec{\Gamma})$ an extra row with the edge gains (as oriented).
   b. The incidence matrix has rank $n-c(\Gamma)$ if Φ is balanced and $n+1-c(\Gamma)$ if Φ is unbalanced.
   c. This is equivalent to a graphic linear program with one linear side condition whose coefficients are the gains.
   d. One can treat several linear side conditions as vector gains. The theory has not been fully worked out yet. (Some work by Geelen et al.)
2. Vectors.
   a. The column of e is a vector $z_e \in F^{n+1}$.
   b. The linear dependencies of vectors z_e are expressed by $L(\Phi)$.
   c. The extra point corresponds to a vector whose extra coordinate is 1, and whose other coordinates are 0. (The same as the vector of a half edge.)
3. Hyperplanes.
   a. The equations $x_j = x_i + \varphi(e{:}v_iv_j)$ determine an affine hyperplane arrangement $\mathcal{A}[\Phi]$ in F^n. This is the *affinographic hyperplane representation* of Φ.
   b. The intersection flats of $\mathcal{A}[\Phi]$ correspond to the balanced flats of $L(\Phi)$.
   c. The intersection flats of the projective completion $\mathcal{A}_{\mathbb{P}}[\Phi]$ correspond to the flats of $L(\Phi)$.

III. Cycle and Cut Spaces; Tensions and Flows

A. *Signed Graphs* (Chen & Wang)

1. Over a field F, with gains in $F^\times$.
   a. Cycle space $Z_1(\Phi; F)$:
      i. It is the null space of the incidence matrix.
      ii. It is the space of F-valued flows (circulations).
      iii. It is generated by characteristic vectors of circuits of the frame matroid $G(\Phi)$.
   b. Cut space $B^1(\Phi; F)$:
      i. It is the row space of the incidence matrix.
      ii. It is the space of F-valued tensions.
      iii. It is generated by characteristic vectors of cocircuits of $G(\Phi)$.
2. Over the integers, with gains in $\mathbb{Z}^+$.

a. Integral cycle lattice $Z_1(\Phi;\mathbb{Z})$:
i. It is the integral null space of the incidence matrix.
ii. It may not contain all integral flows (circulations).
iii. It is generated by characteristic vectors of circuits of the frame matroid $G(\Phi)$ and other strange edge sets. (Chen, Wang, & Zaslavsky)
iv. Nowhere-zero integral k-flows are counted by a polynomial function of k. (Kochol)

b. Integral cut lattice $B^1(\Phi;F)$:
i. It is the row space of the incidence matrix.
ii. It is the space of F-valued tensions.
iii. It is generated by characteristic vectors of cocircuits of $G(\Phi)$.

B. *Gain Graphs:*
Little or no work known to me.

IV. Characteristic and Chromatic Polynomials and Other Invariants

A. *Coloring*
We assume the gain group $\mathfrak{G}$ is finite. Let

$$\mathfrak{C}_k^* := \mathfrak{G} \times [k], \qquad \mathfrak{C}_k := \mathfrak{C}_k^* \cup \{0\}.$$

1. Definitions.
a. A *k-coloration* is a function $f : V \to \mathfrak{C}_k$.
b. It is *zero free* if it maps into $\mathfrak{C}_k^*$.
c. It is *proper* if, for every edge $e{:}vw$, $f(w) \neq f(v)\varphi(e{:}vw)$.
d. An edge is *improper* if $f(w) = f(v)\varphi(e{:}vw)$. The set of improper edges is $I(f)$.
e. The number of proper k-colorations, $\chi_\Phi(k|\mathfrak{G}|+1)$, is called the *chromatic polynomial* of Φ.
f. The number of zero-free proper k-colorations, $\chi_\Phi^{\mathrm{b}}(k|\mathfrak{G}|)$, is called the *zero-free* or *balanced chromatic polynomial* of Φ (depending on the point of view; see further on).

2. Properties.
a. $\chi_\Phi(\lambda)$ is a polynomial, monic of degree n, and otherwise similar to the chromatic polynomial of an ordinary graph. So is $\chi_\Phi^{\mathrm{b}}(\lambda)$.
b. $\chi_\Phi(\lambda)$ equals the characteristic polynomial of the frame matroid, $G(\Phi)$, times a factor of $\lambda^{b(\Phi)}$. Thus it is the characteristic polynomial of $\operatorname{Lat}\Phi$, up to the same factor.
c. $\chi_\Phi(\lambda)$ equals the characteristic polynomial of $\operatorname{Lat}^{\mathrm{b}}\Phi$ times a factor of $\lambda^{c(\Gamma)}$.
d. Algebraic formulas:

$$\chi_\Phi(\lambda) = \sum_{S\subseteq E} \lambda^{b(S)}, \qquad \chi_\Phi^{\mathrm{b}}(\lambda) = \sum_{S \text{ balanced}} \lambda^{b(S)}.$$

(The latter explains the name "balanced chromatic polynomial".)

B. *Extensions*

1. Dichromatic Polynomials.
   a. Combinatorial definitions:
      i. The dichromatic polynomial $Q_\Phi(u,v)$ in the normalized form
      $$\bar{Q}_\Phi(uv,v) := v^{-n} Q_\Phi(u,v)$$
      counts k-colorations by size of the improper edge set:
      $$\bar{Q}_\Phi(k|\mathfrak{G}|+1,v) = \sum_f (v+1)^{|I(f)|}$$
      summed over k-colorations.
      ii. The balanced dichromatic polynomial $Q^{\mathrm{b}}_\Phi(u,v,z)$ in the normalized form
      $$\bar{Q}^{\mathrm{b}}_\Phi(uv,v) := v^{-n} Q^{\mathrm{b}}_\Phi(u,v)$$
      counts zero-free k-colorations by size of the improper edge set:
      $$\bar{Q}^{\mathrm{b}}_\Phi(k|\mathfrak{G}|,v) = \sum_f (v+1)^{|I(f)|}$$
      summed over zero-free k-colorations.
   b. Algebraic definitions:
   $$Q_\Phi(u,v) = v^{-n} \sum_{S \subseteq E} (uv)^{b(S)} v^{|S|}$$
   and $Q^{\mathrm{b}}_\Phi(u,v)$ summed over balanced sets S. These are the graph versions of the corank-nullity (rank generating) polynomial of a matroid or semimatroid (respectively). They generalize Tutte's dichromatic polynomial of a graph.
2. Whitney-Number Polynomials.
   a. Combinatorial definitions:
      i. The Whitney-number polynomial $w_\Phi(x,\lambda)$ counts k-colorations by rank of the improper edge set: $w_\Phi(x, k|\mathfrak{G}|+1) = \sum_f x^{\operatorname{rk} I(f)}$ summed over k-colorations.
      ii. The balanced Whitney-number polynomial $w^{\mathrm{b}}_\Phi(x,\lambda)$ counts zero-free k-colorations by rank of the improper edge set: $w^{\mathrm{b}}_\Phi(x, k|\mathfrak{G}|) = \sum_f x^{\operatorname{rk} I(f)}$ summed over zero-free k-colorations.
   b. Algebraic definitions:
   $$w_\Phi(x,\lambda) = \sum_{R \subseteq S \subseteq E} x^{n-b(R)} \lambda^{b(S)} (-1)^{|S \setminus R|}$$
   and $w^{\mathrm{b}}_\Phi(x,\lambda)$ summed over balanced sets S.
   c. The coefficients of $w_\Phi(x,\lambda)$ are the "doubly indexed" Whitney numbers of the first kind of $\operatorname{Lat}\Phi$.
   d. The coefficients of $w_\Phi(x,-1)$ count faces of $\mathcal{H}[\Phi]$ when $\mathfrak{G} = \mathbb{R}^\times$. (Then the coloring definition can't be used because the group is infinite.)
   e. The coefficients of $w^{\mathrm{b}}_\Phi(x,-1)$ count faces of $\mathcal{A}[\Phi]$ when $\mathfrak{G} = \mathbb{R}^+$.
3. Polychromatic Polynomials.

a. The general polychromatic polynomial combines all the foregoing:

$$q_\Phi(w,x,u,v,z) := \sum_{R\subseteq S\subseteq E} w^{|R|} x^{n-b(R)} \lambda^{b(S)} v^{|S\setminus R|} z^{c(S)-b(S)}.$$

b. The polychromatic polynomial results from setting $z=1$. The balanced polychromatic polynomial results from setting $z=0$, thus restricting the summation to balanced sets R and S.

c. The pairs of polynomials above are obtained by specializing the (balanced) polychromatic polynomials.

C. *Arbitrary Gain Graphs and Biased Graphs*

We assume the graph is finite. The order of the gain group, if any, is immaterial.

1. The algebraic formulas are used to define chromatic and other polynomials.
2. Reduction formulas:
   a. Multiplication: $Q_{\Omega_1 \,\dot\cup\, \Omega_2} = Q_{\Omega_1} Q_{\Omega_2}$.
   b. Deletion-contraction: $Q_\Omega = Q_{\Omega\setminus e} - Q_{\Omega/e}$ if e is a link, and the same for the balanced dichromatic polynomial.
   c. Both polynomials remain the same if balanced loops are changed to loose edges or vice versa.
   d. The balanced dichromatic polynomial remains the same if all unbalanced edges are deleted.
3. Universality: Any function of biased graphs with the first three properties is an evaluation of Q_Ω, and if it has the last property it is an evaluation of Q^{b}_Ω.

V. Oriented Matroids

A. *Signed-Graphic Matroids*

1. Orient Σ with bidirected edges:
   a. Definition: one arrow at each end. Positive edge: arrows agree. Negative edge: arrows conflict.
   b. Direct generalization of orientation of an ordinary graph.
   c. (Bidirected) cycle: an oriented signed-graph circuit with no source or sink.
   d. Acyclic orientations $\leftrightarrow$ regions of $\mathcal{H}[\Sigma]$.
   e. Number of acyclic reorientations $= |\chi_\Sigma(-1)|$ (by oriented matroid theory).
2. Orientations of $G(\Sigma)$: (Slilaty)
   a. Orientation from bidirection.
   b. Other orientations? For an inseparable signed graph with non-binary frame matroid:
      i. No other orientation classes for some.
      ii. At least three orientation classes for most, obtained from circle orientations (see below). Conjecturally, only three.
      iii. Determining factors: $[\pm C_3]$ and $[-K_4]$ minors.
3. Orientations of $L(\Sigma)$: (Slilaty)
   a. For an inseparable signed graph with non-regular lift matroid:

i. Impossible for most.
ii. Unique for the rest.
iii. Determining factors: $[\pm C_3]$ and $[-K_4]$ minors.

b. Note that if $L(\Sigma)$ is regular, it equals $G(\Sigma)$, so the orientations are the same.

B. *Frame Matroids*

1. Biased Graphs.
   a. *Circle orientation*: Circles are oriented, not edges. (Slilaty)
      i. A theta-graph consistency condition.
      ii. Cycles and acyclic orientations are defined.
   b. Signed Biased Graphs. (Slilaty)
      i. Combination of circle orientation and bidirection.
2. Gain Graphs.
   a. Ordered gain group.
   b. Gain group $\mathbb{R}^{\times}$: consistent with regions of $\mathcal{H}[\Phi]$. (Slilaty)

C. *The Balanced Semimatroid*

1. This is the right approach to orienting the lift: not the whole matroid. (Modelled on affinographic hyperplane arrangements.)
2. Conjecturally, there are orientations of the balanced semimatroid that are not restrictions of orientations of the frame matroid.
3. Orientation of a semimatroid has never been defined.

D. *Nonorientable Matroids*

1. Minimal projective, nonorientable matroids are contained in $L_0(\Phi)$, where Φ has gains in $\mathbb{F}_p$. (Flórez & Forge)

VI. Structure and Isomorphism

A. *k-Sums*

1. Defined mainly between a biased graph and a balanced graph.
2. 3-sum along a $\langle \pm K_2^\bullet \rangle$ subgraph.

B. *Isomorphism*

1. Frame Matroids:
   a. In general: not known.
   b. Signed graphs: slightly known.
   c. Bicircular matroids: isomorphism corresponds to complicated graph operations. (D.K. Wagner; Coullard, del Greco, & Wagner)
   d. Group expansions: in progress. (Zaslavsky)
2. Lift Matroids:
   a. In general: not known.
   b. Signed graphs: partly known. (A major part is in progress by Guenin, Pivotto, & Wollan.)
   c. Group expansions: in progress. (Zaslavsky)
3. Frame $\cong$ Lift Matroids:
   a. In general: not known.

b. Group expansions: in progress. (Zaslavsky)

C. *Graph Properties*

1. Disjoint Unbalanced Circles:
   a. Signed graphs with no 2 disjoint unbalanced circles are classified. (Slilaty)
      i. Balanced.
      ii. Balancing vertex.
      iii. Entangled: Projective planar or $[-K_5]$, k-summed with various balanced signed graphs for various $k \leq 3$.
   b. Contrabalanced graphs with no disjoint (unbalanced) circles are classified. (Lovász)
2. Possible Gain Groups:
   a. A biased graph can be signed iff it has no contrabalanced theta subgraph.
   b. A finite biased graph can have gains in an infinite group but not in any finite group. (Brooksbank, Qin, E. Robertson, & Seress)

D. *Matroid Properties*

1. Biased Graphs:
   a. Supersolvability of frame and lift matroids. (Zaslavsky; Koban)
2. Gain Graphs:
   a. The gain graphs with fixed gain group are, like projective spaces with fixed coordinate field, a fundamental class in matroid theory (a "variety" of matroids). (Kahn & Kung)
3. Signed Graphs:
   a. Nonbinary iff no $\pm K_2^\bullet$ minor.
   b. Regular iff no 2 disjoint negative circles iff $G(\Sigma) \neq L(\Sigma)$ (for inseparable Σ).
   c. $\mathbb{F}_4$-representable iff cylindrical, or mesa graph, or no link minor $[\pm C_3]$ or $[-K_4]$, up to 3-summing with balance signed graphs. (Pagano; Gerards & Schrijver)
   d. Assuming 3-connected and no 2 disjoint negative circles: $\mathbb{F}_4$-representable implies $\langle\Sigma\rangle$ has gains in $\mathbb{Z}_3$. (Pagano)
   e. $\mathbb{F}_4$-representable iff nearly poised (i.e., discrepancy ≤ 1). (Gerards & Schrijver; Pagano)

VII. Duality and Embedding

A. *Signed Graphs* (mostly Slilaty)

Graph in surface has signs according to a $\mathbb{Z}_2$-homology rule.

1. Projective Plane:
   a. Embed Γ noncontractibly and let Σ be its signed geometric dual. Then $G^*(\Sigma) = G(\Gamma)$.
   b. Embed Σ and let Γ be its unsigned geometric dual. Then $G^*(\Sigma) = G(\Gamma)$.

c. If Γ is nonplanar and $G^*(\Sigma) = G(\Gamma)$, then Γ and Σ have dual projective-planar embeddings.

2. Torus, Klein bottle, and Cylinder:
   a. Use dual homology rules for signs.
   b. Embed Σ and let Σ^* be its geometric dual. Then their frame matroids are dual. (Assume sufficient connectivity.)
   c. Conjecture: The converse.

B. *Gain Graphs*

1. Some connection between surface duality and matroid duality? (Slilaty)

VIII. Recognition

The main problem is to recognize a matrix whose matroid is a frame matroid.

A. *Real Multiplicative Frame Matroids*

1. Recognition is an important and difficult problem in linear optimization.
2. It contains recognition of bicircular matroids.

B. *Bicircular Matroids* (Chandru, Collard, del Greco, & D.K. Wagner)

1. Recognition of a matrix with bicircular matroid is NP-complete.
2. Deciding whether a real gain graph is contrabalanced is NP-complete.

C. *Signed-Graphic Matroids*

1. Binet matrices are under study. (Appa et al.)
2. Recognizing when the matroid is graphic depends on recognizing certain forbidden minors.
3. Recognizing contrabalance (i.e., when the frame matroid is bicircular) is trivial.

IX. Forbidden Minors

A. *Standard Matroids*

1. $G(K_n)$, $G(K_{3,3})$, F_7, F_7^-, $R_{10} = G(-K_5)$, duals: characterized as $G(\Omega)$, $L(\Omega)$, $L_0(\Omega)$. (Zaslavsky; Slilaty)

B. *Frame Matroids*

1. Many small forbidden minors of rank 3; no large ones.
2. Signed graphs: Regular forbidden minors are known (many). (Qin, Slilaty, & Zhou)
3. Gain graphs: Unknown.
4. Gain graphs over a specific group: Essentially nothing is known.

X. Generalizations

A. *Matroids with Gains*

1. Signed Matroids.
   a. Binary clutters:
      i. Definition: the class $\mathcal{C}_-(M,\sigma)$ of negative circuits.
      ii. Equivalently: a port of a binary matroid.
      iii. $L(M,\sigma)$ and $L_0(M,\sigma)$.

iv. MFMC related to excluding $\mathcal{C}(F_7)$, $\mathcal{C}_-(-K_5)$, $\mathcal{C}_-(-K_4)$ as minors. (Seymour)

v. Ideal if excludes $\mathcal{C}(F_7)$, $\mathcal{C}_-(-K_5)$ and blocker, $\mathcal{C}_-(-K_4)$ and blocker as minors. (Novick & Sebö; Cornuéjols & Guenin)

2. Oriented Matroids with Gains.
   a. Orientation and abelian group required to define gain of a circuit.
   b. Lift matroid of $(\mathcal{M}, \varphi)$ defined via lifting signature. (Koban)

B. *Hypergraphs with Gains*

1. The "gain" of a hyperedge is an equivalence class of functions $V(e) \to \mathfrak{G}$ under the left action of $\mathfrak{G}$. Little is known about matroids.
2. Signed hypergraphs reduced (in part) to signed graphs. (Rusnak)
3. $\mathfrak{G} = \mathbb{F}_q^\times$ gives higher-weight Dowling geometries, associated with error-correcting codes. (Dowling, Bonin)

XI. More References and Information

A. My bibliography [19].

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