Universal and Topological Gains for Biased Graphs

Preliminary Version. To Be Expanded.

by

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Abstract

A *gain graph* is a graph Γ together with a *gain function*, which assigns to each edge a value from a group so that reversing the orientation of an edge inverts its gain. Γ and the class of circles (edge sets of simple closed walks) whose edge gain product equals 1 (called *balanced* circles) constitute a *gain-biased graph*. We show that a gain-biased graph has a canonical gain function, of which any gain function is a homomorphic image; that every gain-biased graph can be embedded in a topological space so that the contractible circles are exactly the balanced ones; and related results.

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Introduction

Suppose we assign to each edge of a graph Γ a *gain*, a value $\varphi(e)$ belonging to a group $\mathfrak{G}$, with the rule that when e is taken in the opposite direction, its gain is inverted; that is, $\varphi(e^{-1}) = \varphi(e)^{-1}$. We call $\Phi = (\Gamma, \varphi, \mathfrak{G})$ a *gain graph*, Γ the *underlying graph*, $\mathfrak{G}$ the *gain group*, and φ the *gain function*. Let $\mathcal{B}(\Phi)$ be the class of circles (edge sets of simple closed walks) in which the product of the edge gains taken in cyclic order is the identity. (Such a circle is called *balanced*.) Then $(\Gamma, \mathcal{B}(\Phi))$ is a *biased graph*, meaning that if two of the circles in a theta subgraph of Γ are balanced, so is the third. A biased graph so derived from a gain graph Φ is called *gain-biased*; we denote it by $[\Phi]$.

In this article we show two things about gain-biased graphs. First, every gain-biased graph has a 'universal' gain function, whose gain group is the free group on the edge set of Γ modulo the relations $C = 1$ for $C \in \mathcal{B}(\Phi)$. (We regard C as a word by taking its edges in cyclic order.) A *topologically biased graph* is a biased graph $(\Gamma, \mathcal{B})$ such that Γ can be embedded in a topological space so a circle is contractible if and only if it is in $\mathcal{B}$. Evidently a topologically biased graph is a kind of gain-biased graph. Our second main result is the converse: every gain-biased graph is topological. (Indeed every gain graph is topological, but that construction is more complicated.) We also develop results, related to the two main theorems, about graphs with arbitrarily distinguished circles.

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1. Preliminaries.

We always denote by Γ a graph (N, E), whose node set is N and edge set is E.[1] We do not assume finiteness. The class of circles of Γ is $\mathfrak{C} = \mathcal{C}(\Gamma)$. A *forest* is a set of edges

[1] Every edge is a link or a loop.

containing no circle. We denote by Φ a gain graph $(\Gamma, \varphi, \mathfrak{G})$, by Ψ a gain graph $(\Gamma, \psi, \mathfrak{H})$, and by Ω a pair $(\Gamma, \mathcal{B})$ where $\mathcal{B} \subseteq \mathcal{C}$ is a class of distinguished circles. In general Ω need not be a biased graph. A gain graph is called *abelian* if its gain group is abelian.

By $\mathfrak{G}$ and $\mathfrak{H}$ we denote groups. If $\mathcal{R} \subseteq \mathfrak{G}$, $\langle \mathcal{R} \rangle$ is the subgroup generated by $\mathcal{R}$. The free group generated by E is $\mathfrak{F}(E)$; the free abelian group generated by E is $\mathfrak{A}(E)$. The notation $\mathfrak{G}/\mathcal{R}$ means the quotient group of $\mathfrak{G}$ by the normal subgroup generated by $\mathcal{R}$, where $\mathcal{R} \subseteq \mathfrak{G}$. A circle in Γ, which strictly speaking is a set of edges, we also regard as a word in $\mathfrak{F}(E)$ in the obvious way; this word is unique up to conjugation and inversion.

Switching Φ means replacing φ by a function φ^λ, where $\lambda : N \to \mathfrak{G}$ is arbitrary, defined by $\varphi^\lambda(e) = \lambda(v)^{-1}\varphi(e)\lambda(w)$, where v and w are the endpoints of e (possibly the same node) and e is oriented from v to w. We say φ and φ^λ, or Φ and $\Phi^\lambda = (\Gamma, \varphi^\lambda, \mathfrak{G})$, are *switching equivalent*, written $\varphi \sim \varphi^\lambda$ or $\Phi \sim \Phi^\lambda$; the equivalence classes of this relation are called *switching classes*. *Conjugating* Φ means switching by a constant function λ.

Gain graphs Φ_1 and Φ_2 are *isomorphic* if there exist isomorphisms $\gamma : \Gamma_1 \to \Gamma_2$ and $g : \mathfrak{G}_1 \to \mathfrak{G}_2$ such that $\varphi_2 = g \circ \varphi_1 \circ \gamma^{-1}$. They are *switching isomorphic* if $\varphi_2 \sim g \circ \varphi_1 \circ \gamma^{-1}$.

A gain graph Φ is *proper* if $\mathfrak{G} = \langle \mathrm{Im}\ \varphi \rangle$. It is *strongly proper* if every gain graph in its switching class is proper. To determine whether or not Φ is strongly proper, it is sufficient to examine one gain graph in its switching class, as we now show.

Proposition 1.1. *A gain graph is strongly proper if and only if it switches to a proper gain graph whose gains are identically* 1 *on some maximal forest of the underlying graph.*

Proof. Suppose Φ has the property that $\varphi|_F \equiv 1$ where F is some maximal forest. We show that every switching Φ^λ is proper; this establishes the result.

First suppose $\lambda =$ constant. Then $\varphi^\lambda(e) = \lambda^{-1}\varphi(e)\lambda$ so $\langle \mathrm{Im}\ \varphi^\lambda \rangle = \lambda^{-1}\mathfrak{G}\lambda = \mathfrak{G}$.

Now let λ be arbitrary. We may suppose Γ is connected. Root it at r and let P_v be the unique path in F from r to v, for each $v \in N$. By conjugating we may take $\lambda(r) = 1$. Then $\varphi^\lambda(P_v) = \lambda(v)$ and, for an edge $e \notin F$ oriented (say) from v to w, we have $\varphi^\lambda(e) = \varphi^\lambda(P_v)^{-1}\varphi(e)\varphi^\lambda(P_w)$. It follows that Im $\varphi^\lambda|_F$ generates Im λ and consequently Im φ^λ generates Im φ. So Φ^λ is proper. ■

2. Universal and canonical gains.

A biased graph that arises from gains has a canonical gain function with a kind of universality. Moreover, given any graph and class of distinguished circles in it, there is a canonical gain function that yields a minimally balanced gain-biased graph in which the given class is balanced.

Given Ω, let $\mathfrak{G}_\Omega = \mathfrak{F}(E)/\mathcal{B}$, let φ_Ω be the natural mapping of E into $\mathfrak{G}_\Omega$, and let $\Phi(\Omega)$ be the gain graph $(\Gamma, \varphi_\Omega, \mathfrak{G}_\Omega)$. We call $\Phi(\Omega)$ the *canonical gain graph* associated with Ω.

We call a gain graph Φ *quasi-universal for* Ω if, whenever Ψ is a gain graph such that $\mathcal{B}(\Psi) \supseteq \mathcal{B}$, then $\psi = h \circ \varphi$ for some homomorphism $h : \mathfrak{G} \to \mathfrak{H}$. If Φ is also proper it is a *universal gain graph for* Ω. Let us partially order all pairs $(\Gamma, \mathcal{B})$, for fixed Γ, by the containment ordering of their classes $\mathcal{B}$. Since $\mathcal{B}(\Psi) \supseteq \mathcal{B}(\Phi)$, $[\Phi]$ is the unique minimal gain-biased graph containing Ω. Notice that the homomorphism h is unique if Φ is universal, since Im φ generates $\mathfrak{G}$ and $h(\varphi(e))$ is thereby determined to be $\psi(e)$.

Theorem 2.1. *Let $\Omega = (\Gamma, \mathcal{B})$, where Γ is a graph and $\mathcal{B} \subseteq \mathcal{C}(\Gamma)$. Then $\Phi(\Omega)$ is a universal gain graph for Ω and every universal gain graph for Ω is isomorphic to $\Phi(\Omega)$. If Ω is gain-biased, then $[\Phi(\Omega)] = \Omega$.*

Proof. Obviously, $\Phi(\Omega)$ is proper. By definition $\mathcal{B}(\Phi(\Omega))$ contains $\mathcal{B}$. Let Ψ be a gain graph for which $\mathcal{B}(\Psi) \supseteq \mathcal{B}$. The gain function $\psi : E \to \mathfrak{H}$ extends to a homomorphism $\bar\psi : \mathfrak{F}(E) \to \mathfrak{H}$ and we have $\mathcal{C} \cap \mathrm{Ker}\ \bar\psi = \mathcal{B}(\Psi) \supseteq \mathcal{B}$. Therefore $\bar\psi$ induces a homomorphism $h : \mathfrak{G}_\Omega \to \mathfrak{H}$ such that $\psi = h \circ \varphi_\Omega$. We conclude that $\Phi(\Omega)$ is a universal gain graph for Ω.

Suppose Φ is also universal for Ω. Then there are homomorphisms $g : \mathfrak{G} \to \mathfrak{G}_\Omega$ and $h : \mathfrak{G}_\Omega \to \mathfrak{G}$ such that $\varphi = h \circ \varphi_\Omega$ and $\varphi_\Omega = g \circ \varphi$. Therefore $\varphi = h \circ g \circ \varphi$ and $\varphi_\Omega = g \circ h \circ \varphi_\Omega$. Since Φ and $\Phi(\Omega)$ are proper, $h \circ g$ and $g \circ h$ are both identity functions. It follows that g and h are inverse isomorphisms under which $\Phi \cong \Phi(\Omega)$.

It is clear that $\mathcal{B}(\Phi(\Omega)) = \mathcal{B}$ if Ω is gain-biased. ■

Supposing Φ in the preceding proof is only quasi-universal for Ω, then $g \circ h$ is the identity on $\mathfrak{G}_\Omega$. It follows that h is a monomorphism and $\mathfrak{G}$ is the semidirect product of Ker g with Im h. Conversely, if a group $\mathfrak{G}$ is the semidirect product of $\mathfrak{G}_0$ and $\mathfrak{G}_\Omega$, and $\Phi = (\Gamma, \varphi, \mathfrak{G})$ is defined by $\varphi = i \circ \varphi_\Omega$ where $i : \mathfrak{G}_\Omega \to \mathfrak{G}$ is the embedding monomorphism, then Φ is clearly quasi-universal for Ω. This shows that the quasi-universal gain graphs for Ω are precisely the isomorphs of those gain graphs obtained from $\Phi(\Omega)$ by enlarging the gain group through the above semidirect product construction.

Because switching classes rather than individual gain graphs are more crucial to balance theory, we may wish to relax the definition of universality. (This will also enable us to shrink the universal gain group.) We call Φ *switching quasi-universal for* Ω if, whenever Ψ is a gain graph such that $\mathcal{B}(\Psi) \supseteq \mathcal{B}$, then $\psi \sim h \circ \varphi$ for some homomorphism $h : \mathfrak{G} \to \mathfrak{H}$, and *switching universal for* Ω if it is also strongly proper. One switching quasi-universal gain graph is $\Phi(\Omega)$.

Given Ω, let F be a forest in Γ. Let $\mathfrak{G}_{\Omega,F} = \mathfrak{F}(E)/(\mathcal{B} \cup F)$ (that is, $\mathfrak{G}_\Omega$ modulo the relations $f = 1$ for all $f \in F$), let $\varphi_{\Omega,F}$ be the natural mapping of E into $\mathfrak{G}_{\Omega,F}$, and let $\Phi_F(\Omega) = (\Gamma,\ \varphi_{\Omega,F},\ \mathfrak{G}_{\Omega,F})$. It is clear that $[\Phi_F(\Omega)] = [\Phi(\Omega)]$. In effect, $\Phi_F(\Omega)$ is the quotient of $\Phi(\Omega)$ by the forest F.

Theorem 2.2. *Let* $\Omega = (\Gamma, \mathcal{B})$, *where* Γ *is a graph and* $\mathcal{B} \subseteq \mathcal{C}(\Gamma)$. *If* F *is a maximal forest in* Γ, *then* $\Phi_F(\Omega)$ *is a switching-universal gain graph for* Ω, *and every switching universal gain graph for* Ω *is switching isomorphic to* $\Phi_F(\Omega)$. *If* Ω *is gain-biased, then* $[\Phi_F(\Omega)] = \Omega$.

Proof. We repeat the proof of Theorem 2.1, but noting that we may assume $\psi|_F \equiv 1$ since any gain function can be switched into this form. ■

Much as with quasi-universal gain graphs, every switching-quasi-universal gain graph for Ω is obtainable (up to switching isomorphism) from any $\Phi_F(\Omega)$ by taking a semi-direct product. This applies in particular to $\Phi(\Omega)$.

It follows from Theorem 2.2 that all $\Phi_F(\Omega)$ for different maximal forests F are switching isomorphic. They are in a sense almost switching equivalent to each other and to $\Phi(\Omega)$. Let φ_Ω^F be φ_Ω switched so it is the identity on F (φ_Ω^F is unique up to conjugation in each connected component of Ω), and extend it to a homomorphism $\mathfrak{G}_\Omega \to \mathfrak{G}_\Omega$. This homomorphism induces a monomorphism $\mu_F : \mathfrak{G}_{\Omega,F} \to \mathfrak{G}_\Omega$ which satisfies $\varphi_\Omega^F = \mu_F \circ \varphi_{\Omega,F}$. Thus μ_F is the homomorphism h appearing in the definition of switching universality when Ψ is taken to be $\Phi(\Omega)$. If we regard $\varphi_{\Omega,F}$ and φ_Ω^F as essentially the same, differing only in that the former has gain group reduced from $\mathfrak{G}_\Omega$ to $\mathfrak{G}_{\Omega,F}$ so as to make $\Phi_F(\Omega)$ proper, then we may say the $\varphi_{\Omega,F}$ are essentially all switching equivalent to each other and to φ_Ω.

Similar results hold for abelian gain graphs. Given Ω, let $\mathfrak{A}_\Omega = \mathfrak{A}(E)/\mathcal{B}$, let α_Ω be the natural mapping $E \to \mathfrak{A}_\Omega$, and let $A(\Omega)$ be the gain graph $(\Gamma, \alpha_\Omega, \mathfrak{A}_\Omega)$, called the *canonical abelian gain graph* associated with Ω.

An abelian gain graph Φ is *abelian quasi-universal for* Ω if it satisfies quasi-universality (as defined earlier) when Ψ is restricted to being abelian. If Φ is proper it is a *universal abelian gain graph for* Ω. Since $A(\Omega)$ is merely $\Phi(\Omega)$ with abelianized gain group, we have

Corollary 2.3. *Let $\Omega(\Gamma, \mathcal{B})$, where Γ is a graph and $\mathcal{B} \subseteq \mathcal{C}(\Gamma)$. Then $A(\Omega)$ is a universal abelian gain graph for Ω and every universal abelian gain graph for Ω is isomorphic to $A(\Omega)$. If Ω is gain-biased by abelian gains, then $[A(\Omega)] = \Omega$.* ■

The abelian quasi-universal gain graphs Φ are precisely the gain graphs obtained from $A(\Omega)$ by enlarging the gain group $\mathfrak{A}_\Omega$ to a direct product $\mathfrak{A}_\Omega \times \mathfrak{A}_0$ with an abelian group

$\mathfrak{A}_0$. This follows from the remark after Theorem 2.1.

We call Φ *abelian switching-quasi-universal for* Ω if it satisfies switching quasi-universality when Ψ is restricted to be abelian. (One example is $A(\Omega)$.) If Φ is strongly proper it is a *switching-universal abelian gain graph for* Ω (or *abelian switching-universal for* Ω). If we abelianize $\mathfrak{G}_{\Omega,F}$ we get a group $\mathfrak{A}_{\Omega,F}$ and gain graph $A_F(\Omega)$. From Theorem 2.2 we have

Corollary 2.4. *Let* $\Omega = (\Gamma, \mathcal{B})$, *where* Γ *is a graph and* $\mathcal{B} \subseteq \mathcal{C}(\Gamma)$. *If* F *is a maximal forest in* Γ, *then* $A_F(\Omega)$ *is a switching-universal abelian gain graph for* Ω, *and every switching-universal abelian gain graph for* Ω *is switching isomorphic to* $A_F(\Omega)$. *If* Ω *is abelian gain-biased, then* $[A_F(\Omega)] = \Omega$. ■

Again, any switching-quasi-universal abelian gain graph is obtained (up to isomorphism) from any $A_F(\Omega)$ by taking a direct product; this applies in particular to $A(\Omega)$.

3. Topological gains.

All gain-biased graphs can be obtained by topological means. The first thing to note is that embedding a graph yields gains.

Theorem 3.1. *Let* Γ *be a graph embedded in a topological space* X. *Let* $\mathcal{B}_X$ *be the class of circles of* Γ *that are contractible in* X *and* $\mathcal{N}_X$ *the class of circles which are homologically null. Then* $(\Gamma, \mathcal{B}_X)$ *is gain-biased and* $(\Gamma, \mathcal{N}_X)$ *is abelian gain-biased.*

Proof. We assume X is pathwise connected; if it is not, we may work separately in each path component. We describe the construction of a gain graph $\Psi_F(\Gamma, X)$ for $(\Gamma, \mathcal{B}_X)$, from which the theorem follows since the homology group $H_1(X)$ is the abelianized fundamental group.

Let F be a maximal forest in Γ. In each component F_i (indexed by a set I) let r_i be a root node. For definiteness say that $0 \in I$. Our gain group will be the fundamental group $\pi_1(X, r_0)$. Let Q_i, for $i \in I$, be a topological path from r_0 to r_i; and let R_v, for $v \in N$, be the concatenation of Q_i with the unique simple walk in F from r_i to v (regarded as a topological path). (We assume Q_0 is the constant path.) We assign to $f \in F$ the gain $\varphi(f) = 1$ and to $e \notin F$ the gain $\varphi(e) = [R_v e R_w^{-1}]$, where e is oriented from v to w (these being its endpoints) and $[P]$ denotes the homotopy class of a topological path P. This defines $\Psi_F(\Omega, X)$.

It is easy to see that $C \in \mathcal{B}_X$ if and only if C has gain 1. ■

We call $\Psi_F(\Gamma, X)$ a *topological gain graph* and $(\Gamma, \mathcal{B}_X)$ a *topologically biased graph.* Abelianizing $\Psi_F(\Gamma, X)$ yields a gain graph $\Xi_F(\Gamma, X)$ whose gain group is $H_1(X)$; the gains are $\chi_F(e) = 0$ for $e \in F$, $\chi_F(e) =$ homology class of $R_v e R_w^{-1}$ for $e \notin F$. We call $\Xi_F(\Gamma, X)$ a *homology gain graph* and $(\Gamma, \mathcal{N}_X)$ a *homologically biased graph.* Theorem 3.1 says that topologically and homologically biased graphs are examples of gain-biased graphs. The main result of this section is the converse.

Theorem 3.2. *Every gain-biased graph is topologically biased. Every abelian gain-biased graph is homologically biased.*

This is a corollary of a more general statement. By attaching a 2-cell *simply* to a topological space we mean attaching a disk by means of a homeomorphism from the boundary into the space. We regard a graph as defining a topological space in the obvious way.

Proposition 3.3. *Let $\Omega = (\Gamma, \mathcal{B})$, where Γ is a graph and $\mathcal{B} \subseteq \mathcal{C}(\Gamma)$. Let $X(\Omega)$ be the topological space obtained by attaching a 2-cell simply to each circle in $\mathcal{B}$. Then $(\Gamma,\ \mathcal{B}_{X(\Omega)}) = [\Phi(\Omega)]$. Indeed, $\Psi_F(\Gamma, X(\Omega)) \cong \Phi_F(\Omega)$.*

Proof. It suffices to prove the last assertion. We may assume Γ is connected.

Let P denote a walk in Γ. A *combinatorial homotopy* is a sequence of transformations by the following two operations and their inverses.

(H1) If $P = Qee^{-1}R$, replace P by QR.

(H2) If $P = QCR$, where C is a simple closed walk along a balanced circle, replace P by QR.

Let $[P]$ denote the combinatorial-homotopy equivalence class of P. For $r \in N$, the group $\pi(\Omega, r)$ of combinatorial homotopy classes of walks from r to r (where multiplication is concatenation) is generated by the classes $[P_v e P_w^{-1}]$ for $e \notin F$, where e has endpoints v and w and is oriented from v to w.

If a walk in Γ is regarded as an element of $\mathfrak{F}(E)$, a combinatorial homotopy class is an element of $\mathfrak{G}_\Omega$. Thus, we may define $\psi : E \to \mathfrak{G}_\Omega$ by $\psi(e) = [P_v e P_w^{-1}]$. Evidently $\psi \sim \varphi_\Omega$; since $\psi|_F \equiv 1$, ψ is a possible choice for φ_Ω^F. It follows that $\pi(\Omega, r) = \mathrm{Im}\, \mu_F$, where $\mu_F : \mathfrak{G}_{\Omega,F} \to \mathfrak{G}_\Omega$ is the monomorphism associated to $\varphi_\Omega^F = \psi$. Therefore $\Psi_F(\Omega, r) = (\Gamma, \psi, \pi(\Omega, r))$ is isomorphic via μ_F^{-1} to $\Phi_F(\Omega)$.

The obvious way of regarding a walk as a topological path yields an isomorphism of $\pi(\Omega, r)$ with $\pi(X(\Omega), r)$. This follows directly from the analogous statement about simplicial complexes (see for instance [1, Theorem 6.3.2] or [2, Section 3.6]). Thus, $\Psi_F(\Gamma, X(\Omega))$ is naturally isomorphic to $\Psi_F(\Omega, r)$. But we have observed the isomorphism of the latter to $\Phi_F(\Omega)$. Thus, we are done. ■

Proof of Theorem 3.2. The first part is obvious from Proposition 3.3. The second part follows because $A_F(\Omega)$ and $\Xi_F(\Gamma, X(\Omega))$ are just the commutative versions of $\Phi_F(\Omega)$ and $\Psi_F(\Gamma, X(\Omega))$. ■

Proposition 3.4. *Any gain graph is switching isomorphic to a topological gain graph. Any abelian gain graph is switching isomorphic to a homological gain graph.*

Sketch of Proof. One can attach 2-cells to the gain graph Φ (if Φ is connected and strictly proper) in order to make a space $X(\Phi)$ whose fundamental group is $\mathfrak{G}$ and such that $\Phi \sim \Phi_F(\Gamma, X(\Phi))$, where F is a spanning tree. The technique is essentially the one used to prove that every group is a fundamental group; cf. [1, p. 245] or [2, Theorem 3.8.10 *et seq.*]. If Φ is disconnected or not strongly proper, one replaces it by a suitable extension graph.

The abelian result is immediate from the non-abelian one. ■

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