Matroid Theory and Its Applications
(Proc. Session of C.I.M.E., Varenna, Italy, 1980)
ed. Adriano Barlotti, pp. 417–423. Ligouri Editore, Naples, 1982.[1]

Voltage-Graphic Matroids

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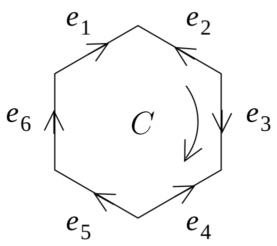
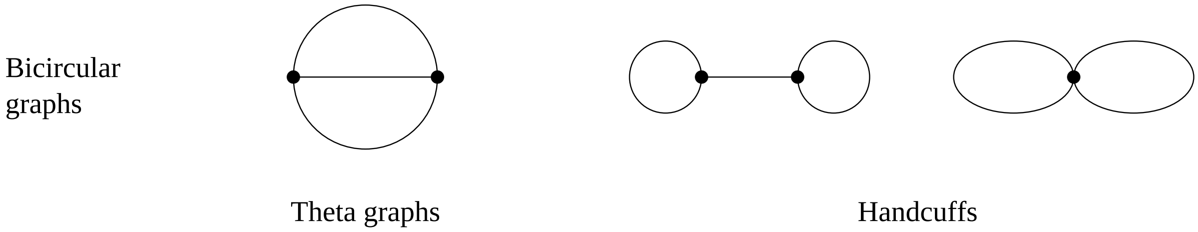
1. A *voltage graph* [now called a *gain graph*[2]] is a pair $\Phi = (\Gamma, \varphi)$ consisting of a graph $\Gamma = (N, E)$ and a *voltage* [now *gain function*], a mapping $\varphi : E \to \mathfrak{G}$ where $\mathfrak{G}$ is a group called the *voltage group* [now *gain group*]. The voltage [gain] on an edge depends on the sense in which the edge is traversed: if for e in one direction the voltage is $\varphi(e)$, then in the opposite direction it is $\varphi(e)^{-1}$. The voltage [gain] on a circle is the product of the edge voltages taken in order with consistent direction; if the product equals 1 the circle is called *balanced*. (While in general the starting point and orientation of C influence its voltage, they have no effect on whether it is balanced.) A subgraph is balanced if every circle in it is balanced. Assuming N is finite, let $n = |N|$ and, for $S \subseteq E$, let $b(S) =$ the number of balanced components of (N, S).



$$\varphi(C) = \varphi(e_1)\varphi(e_2)^{-1}\varphi(e_3)\varphi(e_4)^{-1}\varphi(e_5)\varphi(e_6)$$

Matroid Theorem. *The function* $\operatorname{rk} S = n - b(S)$ *is the rank function of a matroid* $G(\Phi)$ *on the set* E. *A set* $A \subseteq E$ *is closed iff every edge* $e \notin A$ *has an endpoint in a balanced component of* (N, A) *but does not combine with edges in* A *to form a balanced circle. A set is a circuit iff it is a balanced circle or a bicircular graph containing no balanced circle.*



[1]This is the original article, reset in LaTeX, with updated terminology and references in square brackets, June, 2006.

[2][The name was changed to avoid confusion with Kirchhoff's voltage laws.]

We call $G(\Phi)$ a *voltage-graphic matroid.* [Now, a *frame matroid* or *gain-graphic bias matroid.*] When it is a simple matroid, it is a subgeometry of the Dowling geometry $Q_n(\mathfrak{G})$.

EXAMPLES

1) $G(\Gamma)$, the *graphic (polygon) matroid*: $\mathfrak{G} = \{1\}$, $\varphi \equiv 1$.

2) Matroids of *signed graphs* Σ: $\mathfrak{G} = \{+1, -1\}$.

3) $EC(\Gamma)$, the *even-cycle matroid* (M. Doob, Tutte): $\mathfrak{G} = \{+1, -1\}$, $\varphi \equiv -1$.

4) $B(\Gamma)$, the *bicircular matroid* (Simões-Pereira, Klee): $\mathfrak{G} = \mathbb{Z}_2^E$, $\varphi(e) = e$; or $\mathfrak{G} =$ the free abelian group on E, $\varphi(e) = e$.

5) $B(\Gamma^\circ)$, $\Gamma^\circ = \Gamma$ with a loop at every node. The lattice of flats is the set of spanning forests of Γ.

6) $ED(\vec{\Gamma})$, the equidirected circle matroid of a digraph $\vec{\Gamma}$ (Matthews): $\mathfrak{G} = \mathbb{Z}$, $\varphi(e) = +1$ when e is taken in the direction assigned by $\vec{\Gamma}$. (Similarly one has $ED_n(\vec{\Gamma})$, the equidirected circle matroid modulo n, when $\mathfrak{G}\mathbb{Z}_n$.)

7) $A(\vec{\Gamma})$, the anticoherent cycle matroid of $\vec{\Gamma}$ (Matthews): $\mathfrak{G} =$ the free group on N, $\varphi(e) = vw$ if e is directed $v \to w$.

8) $\Phi = \mathfrak{G} \cdot \Delta$, $\Delta =$ a graph on n nodes; Φ is Δ with each edge replaced by every possible $\mathfrak{G}$-labeled edge.

9) $Q_n(\mathfrak{G})$, the Dowling geometry of rank n of $\mathfrak{G}$, is $G(\mathfrak{G} \cdot K_n^\circ)$.

2. Now let $\mathfrak{G}$ have finite order g. A *proper μ-coloring* of Φ is a mapping

$$\kappa : N \to \{0\} \cup (\{1, \dots, \mu) \times \mathfrak{G})$$

such that, for any edge e from v to w (including loops), we have $\kappa(v) \neq 0$ or $\kappa(w) \neq 0$ and also

$$\kappa_1(v) \neq \kappa_1(w) \quad \text{or} \quad \kappa_2(w) \neq \kappa_2(v)\varphi(e) \quad \text{if} \quad \kappa(v), \kappa(w) \neq 0,$$

where κ_1 and κ_2 are the numerical and group parts of κ. Let $\chi_\Phi(\mu g + 1) =$ the number of proper μ-colorings of Φ and let $\chi_\Phi^{\mathrm{b}}(\mu g) =$ the number which do not take the value 0.

Chromatic Polynomial Theorem. *$\chi_\Phi(\mu g + 1)$ is a polynomial in μ. Indeed $\chi_\Phi(\lambda) = \lambda^{b(E)} p(\lambda)$, where $p(\lambda)$ is the characteristic polynomial of $G(\Phi)$.*

Balanced Chromatic Polynomial Theorem. *$\chi_\Phi^{\mathrm{b}}(\mu g)$ is a polynomial in μ. Indeed $\chi_\Phi^{\mathrm{b}}(\lambda) = \sum_A \mu(\emptyset, A)\lambda^{b(A)}$, summed over balanced flats $A \subseteq E$.*

Fundamental Theorem. *Let $\chi_X^{\mathrm{b}}(\lambda)$ denote the balanced chromatic polynomial of the induced voltage graph on $X \subseteq N$. Then*

$$\chi_\Phi(\lambda) = \sum_{X \text{ stable}} \chi_X^{\mathrm{b}}(\lambda - 1).$$

This theorem reduces calculation of $\chi_\Phi(\lambda)$, or of $p(\lambda)$, to that of $\chi^{\mathrm{b}}_\Phi(\lambda)$, which is often easy.

EXAMPLES (continued)

1) $\chi^{\mathrm{b}}_\Phi(\lambda) = \chi_\Gamma(\lambda)$.

4) $\chi^{\mathrm{b}}_\Phi(\lambda) = \sum_k (-1)^{n-k} f_k \lambda^k$, where $f_k =$ the number of k-tree spanning forests in Γ.

3) $\chi^{\mathrm{b}}_\Phi(\lambda) = \sum_A 2^{n-\mathrm{rk}\,A} \chi_{\Gamma/A}(\lambda/2)$, summed over flats A of $G(\Gamma)$.

8) $\chi^{\mathrm{b}}_\Phi(\lambda) = g^n \chi_\Delta(\lambda/g)$.

9) $p(Q_n(\mathfrak{G});\lambda) = g^n((\lambda-1)/g)_n$, where $(x)_n$ is the falling factorial.

3. There is a geometric realization when $\mathfrak{G} \subseteq \mathbb{R}^X$. Let $\mathcal{H}[\Phi]$ be the set of all hyperplanes $x_j = \varphi(e)x_i$ in $\mathbb{R}^n$ where $e \in E$ is an edge from v_i to v_j.

Representation Theorem. *The lattice of all intersections of subsets of $\mathcal{H}[\Phi]$, ordered by reverse inclusion, is isomorphic to the lattice of flats of $G(\Phi)$.*

Corollary. *$\mathcal{H}[\Phi]$ cuts $\mathbb{R}^n$ into $|\chi_\Phi(-1)|$ regions (n-dimensional cells).*

4. Each Φ has a covering graph $\tilde{\Phi} = (\mathfrak{G} \times N, \mathfrak{G} \times E)$, an unlabelled graph. If e goes from v to w, the covering edge (g, e) extends from (g, v) to $(g\varphi(e), w)$. Let $p : \mathfrak{G} \times E \to E$ be the covering projection.

Covering Theorem. *A set $S \subseteq E$ is closed in $G(\Phi)$ iff $p^{-1}(S)$ is closed in $G(\tilde{\Phi})$.*

5. The Matroid Theorem does not essentially require a voltage. All we need is a specified class of "balanced" circles in Γ, such that if two circles in a theta graph are balanced, then the third is also. The pair $(\Gamma, \mathcal{B})$ is a *biased graph.* Although a biased graph cannot be colored in the usual sense, it has algebraically defined "chromatic polynomials" that satisfy the Fundamental Theorem.

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