What About the Chromatic Zeros of a Signed Graph?

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A *signed graph* Σ is a graph $|\Sigma|$ with a sign function on the edges, $\sigma : E \to \{+, -\}$. A (simple) cycle is called positive or negative depending on the product of its edge signs, and Σ is *balanced* if every cycle is positive; for instance, when all edges are positive. To (properly) k-colour a signed graph, use the colour set $C_k = \{0, \pm 1, \ldots, \pm k\}$ or the zero-free colour set $C_k^* = \{\pm 1, \ldots, \pm k\}$ with the rule that the colours x_i, x_j of two vertices joined by a positive edge satisfy $x_i \neq x_j$, while the colours of vertices joined by a negative edge satisfy $x_i \neq -x_j$. (The two vertices are the same when the edge is a loop.) See [2].

Define $\chi_\Sigma(2k+1)$ = the number of k-colourings and $\chi^*_\Sigma(2k)$ = the number of zero-free k-colourings. These functions are monic polynomials $\chi_\Sigma(q)$ and $\chi^*_\Sigma(q)$ of degree $n = |V|$; they are called the *chromatic polynomial* and the *zero-free chromatic polynomial* of Σ. They are equal if Σ is balanced and then they equal $\chi_{|\Sigma|}(q)$, the chromatic polynomial of the underlying graph; thus, an unsigned graph is equivalent to an all-positive signed graph. But otherwise the two chromatic polynomials are not equal.

The Problem. Explore their zeros. Nothing is known. For example, is there any connection between limit curves of zeros of the two polynomials, for interesting graph classes?

Some signed graphs based on an unsigned graph G are $-G$, the all-negative signed graph; $\pm G$, the signed graph that has all edges of G with both positive and negative signs; and either of these with a negative loop at every node, denoted $-G^\circ$ and $\pm G^\circ$. Formulas of sorts for their chromatic polynomials are in [3], e.g., $\chi_{\Sigma^\circ}(q) = \chi^*_\Sigma(q-1)$ and $\chi^*_{\pm G}(q) = 2^n \chi_G(q/2)$.

I know two relationships between the ordinary and zero-free chromatic polynomials. First,

$$\chi_\Sigma(q) = \sum_{X \text{stable}} \chi^*_{\Sigma - X}(q-1),$$

from [3, (1.1)]. Second, χ_Σ and χ^*_Σ are the two constituent polynomials of an Ehrhart quasipolynomial of period two (or one, if balanced) [1, Section 5]. I also know that χ_Σ is essentially the characteristic polynomial of a matroid associated with Σ, a generalisation of the similar relationship for unsigned graphs, but χ^*_Σ is not.

References

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[2] Thomas Zaslavsky, Signed graph coloring. *Discrete Math.* **39** (1982), 215–228. MR 84h:05050a. Zbl. 487.05027.

[3] Thomas Zaslavsky, Chromatic invariants of signed graphs. *Discrete Math.* **42** (1982), 287–312. MR 84h:05050b. Zbl. 498.05030.